1 2.5: Continuity

Definitions:

\( f \) is continuous at \( x = a \) if and only if
\[
\lim_{x \to a} f(x) = f(a)
\]
(i.e., the function approaches the y-value)

3 implications:

1) \( f(a) \) exists (\( f \) defined at \( x = a \))

2) \( \lim_{x \to a} f(x) \) exists

3) \( \text{limit} = f(a) \)

Removable Discontinuities

\( f \) has a removable discontinuity at \( x = a \) if there is a continuous function \( g \) such that \( g(x) = f(x) \) if \( x \neq a \).

\( \text{(i.e. } \lim_{x \to a} f(x) \text{ exists) } \) \( g(a) = \text{limit} \)
Theorems:

Limits inside Continuous Functions

\[
\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right)
\]

Continuity of Polynomial/Rational Functions

- Polynomials are continuous on \((-\infty, \infty)\)
- Rational functions are continuous on their domains

Intermediate Value Theorem

If \(f\) is continuous on \([a, b]\) and \(N\) is between \(f(a)\) and \(f(b)\), then there is a \(c\) between \(a\) and \(b\) such that \(f(c) = N\).

\(x = c\) is a solution to \(f(x) = N\)
Warm-Up

Examples:

If \( f(x) = \begin{cases} 1 - x & \text{if } x \geq 1 \\ -x & \text{if } x < 1 \end{cases} \)

determine whether \( f \) is continuous at \( x = 1 \) or not. Explain your answer precisely. Is \( f \) continuous from the left or right? Does \( f \) have a removable discontinuity?

1. \( f(1) = 1 - 1 = 0 \) \( f(1) \) is defined

2. \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -x = -1 \)

3. \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 1 - x = 0 \)

\( f \) is not cts at \( x = 1 \) because limit DNE
\( f \) is not cts from left because left hand limit \( \neq f(1) \)
\( f \) is cts from right because right hand limit = \( f(1) \)
\( f \) does not have a removable discontinuity because limit DNE
\[
\lim_{x \to 1} \sqrt{\frac{x^2 + 2x - 3}{x - 1}} = \\
= \sqrt{\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1}} = \\
= \sqrt{\lim_{x \to 1} \frac{(x+3)(x-1)}{x-1}} = \\
= \sqrt{4} = 2
\]

Is there a real solution to the equation \(x^5 - x^2 + 2x - 6 = 0\)? If so, find the value of \(a\) such there is a solution in the interval \([a, a + 1]\).

**IVT:** If \(f\) cts on \([a, b]\) and \(N\) is between \(f(a)\) and \(f(b)\), then there is a solution to \(f(x) = N\) between \(a\) and \(b\).

- \(f(x) = x^5 - x^2 + 2x\)
- \(a = 0\)
- \(b = 2\)
- \(N = 6\)

\(f(0) = 0 < 6\) and \(f(2) = 32 > 6\)

\(\square\) \(f(x) = x^5 - x^2 + 2x\) is cts on \([0, 2]\) since \(f\) is a polynomial

\(\square\) \(0 < b < 32\) so \(f(a) < N < f(b)\)

\(\square\) there is a solution to \(x^5 - x^2 + 2x = 6\) between 0 and 2 by IVT

\(f(1) = 1-1+2 = 2\) so there is a solution between 1 and 2, so \(a = 1\)
On Beyond Average: Let $A$ be a constant, and consider the function

$$f(x) = \begin{cases} 
4x + A & \text{if } x < 2 \\
2 & \text{if } x = 2 \\
x^2 - Ax + 1 & \text{if } x > 2
\end{cases}$$

Determine the value of $A$ for which $\lim_{x \to 2} f(x)$ exists or explain why there is no such value $A$.

$$\begin{align*}
\lim_{x \to 2^-} f(x) &= \lim_{x \to 2} \frac{4x + A}{x + 2} = 8 + A \\
\lim_{x \to 2^+} f(x) &= \lim_{x \to 2} \frac{x^2 - Ax + 1}{x + 2} = 4 - 2A + 1
\end{align*}$$

For $\lim_{x \to 2} f(x)$ to exist, $8 + A = 4 - 2A + 1$.

Solving for $A$:

$$8 + A = 4 - 2A + 1$$

$$3A = -3$$

$$A = -1$$

So, $A = -1$ for the limit to exist at $x = 2$. Otherwise, since $\lim_{x \to 2} f(x) \neq f(2)$, $f$ is not continuous at $x = 2$. 