1  3.1: The Derivative

Now that we can find the slope of the line tangent to a curve at any point (provided the limit of the slope exists), we can talk about a new function based on this calculation.

**Definition:** The derivative function of a function \( f \) (or the derivative of \( f \)) is a function defined by

\[
f'(x) = \lim_{{h \to 0}} \frac{f(x+h)-f(x)}{h}
\]
provided the limit exists.

When is \( f \) not differentiable? (i.e., when does \( f'(x) \) not exist or when is \( x \) not in the domain of \( f' \)?

1) **limit is infinite** \( \rightarrow \) tangent line is vertical

2) left hand limit \( \neq \) right hand limit \( \rightarrow \) sharp corner

3) \( f \) not continuous
Warm-Up

Examples:

Find the derivative of \( f(x) = x^2 - 3x \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 - 3(x+h)] + (x^2 + 3x)}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 - 3x}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h} = \lim_{h \to 0} \frac{x(2x + h - 3)}{h} = 2x - 3
\]
Examples:

Let \( f(x) = \frac{8}{x+2} \). Find \( f'(x) \) and use it to determine the slope of the line tangent to \( f \) at the point where \( x = 0 \), \( x = 2 \) and \( x = -1 \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]
\[
= \lim_{h \to 0} \frac{\frac{8}{(x+h)+2} - \frac{8}{x+2}}{h}
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{\frac{8}{(x+h)+2} - \frac{8}{x+2}}{(x+h)(x+2)} \right)
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{\frac{8x+16-8x-8h-16h}{(x+h)(x+2)}}{(x+h)(x+2)} \right)
\]
\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{-8h}{(x+h)(x+2)} \right) = -\frac{8}{(x+2)^2}
\]

So \( f'(0) = -\frac{8}{(0+2)^2} = -\frac{8}{4} = -2 \)

\( f'(2) = -\frac{8}{(2+2)^2} = -\frac{8}{16} = -\frac{1}{2} \)

And \( f'(-1) = -\frac{8}{(-1+2)^2} = -\frac{8}{1} = -8 \)
On Beyond Average: A clock has a radius of 10 cm. Let \( f(t) \) be the horizontal position of the tip of the second hand (where \( f(t) = 0 \) refers to the diameter through the 12 and 6). Assuming the hand starts at the 12, sketch a rough graph of \( f(t) \), then sketch the graph of \( f'(t) \).
Determine whether \( f(x) = |x^2 - 9| \) is differentiable at \( x = 3 \).

**Method I:**

\[
   f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}
\]

\[
   \begin{cases}
   -(x^2-9) & \text{if } x^2-9 < 0 \\
   x^2-9 & \text{if } x^2-9 \geq 0
   \end{cases}
\]

\[
   f(x) = |x^2-9|
\]

**Sketch graph**

- Graph of \( f \)
- \( f'(3) \) does not exist
  - \( f \) is not differentiable at \( x = 3 \)
  - Since there is a sharp corner;
  - i.e., limit of the slopes is different from left and right.