5.2: Maxima and Minima

Definitions:

$f$ has a relative maximum at $x = a$ if and only if $f(a) > f(x)$ for all $x$ "near" $a$.

$f$ has a relative minimum at $x = a$ if and only if $f(a) < f(x)$ for all $x$ "near" $a$.

Fermat’s Theorem: If $f$ has a relative maximum or relative minimum at $x = a$ and $f$ is differentiable at $x = a$, then $f'(a) = 0$.

Note: $f'(a) = 0$ does NOT guarantee a rel max/min. For $f(x) = x^3$, $f'(0) = 3(0) = 0$ but no rel max or min.
More Definitions:

\( f \) has a **critical value** at \( x = a \) if and only if 
\[ f'(a) = 0 \text{ or } f'(a) \text{ DNE} \]

\( f \) has an **absolute maximum** at \( x = a \) if and only if
\[ f(a) \geq f(x) \text{ for all } x \]

in the domain (general or restricted)

\( f \) has an **absolute minimum** at \( x = a \) if and only if
\[ f(a) \leq f(x) \text{ for all } x \]

in the domain (general or restricted)

**Extreme Value Theorem** If \( f \) is continuous on a closed, bounded interval, then

\( f \) is guaranteed to attain its absolute maximum and absolute minimum on the interval.

Graphical examples to show that each of the conditions must hold to guarantee the conclusion:

- **Case 1:** \( f \) not continuous
  - No absolute max
  - \( \text{abs min} = 0 \text{ at } x = b \)

- **Case 2:** \( f \) continuous, not closed interval
  - \( \text{abs max} = 4 \text{ at Critical Value} \)
  - \( \text{No abs min} \)
Examples:

Find the absolute maximum and absolute minimum of \( f(x) = 4x^3 - 15x^2 + 12x + 7 \) on the interval \( 0 \leq x \leq 3 \).

Critical Values:

\[
\begin{align*}
\text{Critical Values:} & \quad f'(x) = 12x^2 - 30x + 12 = 0 \\
& \quad 6(2x^2 - 5x + 2) = 0 \\
& \quad 6(2x-1)(x-2) = 0
\end{align*}
\]

Closed bounded interval

Candidates:

\[
\begin{align*}
f\left(\frac{1}{2}\right) &= \frac{39}{4} = 9.75 \\
f(2) &= 3 \\
f(0) &= 7 \\
f(3) &= 16
\end{align*}
\]

Abs max is 16 when \( x = 3 \)

Abs min is 3 when \( x = 2 \)
Warm Up

Find the absolute maximum and absolute minimum of \( f(x) = \frac{\ln x}{x^2} \) on the interval \((0, 3)\).

Critical Values:

\[ f'(x) = \frac{x^2 \left( \frac{1}{x} \right) - \ln x \cdot (2x)}{x^4} = 0 \]

\[ \frac{x - 2x \ln x}{x^4} = 0 \]

\[ x \left( 1 - 2 \ln x \right) = 0 \]

1. \(2 \ln x = 0\)
2. \(-2 \ln x = -1\)

\[ \frac{\ln x}{x^4} = \frac{1}{e^2} \quad \text{for} \quad x = e^{\frac{1}{2}} \]

\( f \) has rel max at \( x = e^{\frac{1}{2}} \)

and \( f \) has abs max at \( x = e^{\frac{1}{2}} \) (since only one change of direction)

Abs max = \( f \left( e^{\frac{1}{2}} \right) = \frac{\ln e^{\frac{1}{2}}}{\left( e^{\frac{1}{2}} \right)^2} = \frac{1}{2e} \)

No abs min

\[ f(0) = \frac{1 - 2 \ln 1}{1^3} > 0 \quad f(3) = \frac{1 - 2 \ln 3}{3^3} < 0 \]
On Beyond Average: Suppose $f$ is differentiable for all $x$, and that $f$ has exactly one critical value at $x = 1$. Find all the critical values of $h(x) = f(x^2)$.