1 6.1-6.2: Area

Introduction:

The second historical problem of Calculus involves finding the area under the graph of a function. Suppose we want to find the area under the graph of \( f(x) = e^{-x^2} \) between \( x = 0 \) and \( x = 2 \) (denoted by \( \int_{0}^{2} e^{-x^2} \, dx \)). We cannot find the exact area geometrically (or even using techniques you may have previously learned in Calculus!), so we approximate the area using rectangles (see figures in class):

\[
\text{Area (rect)} = \left[ 0.5 \right] f(0) + 0.5 \cdot f(0.5) + 0.5 \cdot f(1) + 0.5 \cdot f(1.5) = \sum_{i=0}^{3} 0.5 \cdot f(0.5i)
\]

\[
\approx 1.1260
\]
What happens if more rectangles are used? \[\text{approximation gets more accurate}\] \[\text{rectangle area \to exact area}\]

How many rectangles are needed to get the exact area? \[\infty \text{ (in most cases)}\]
Definitions:

**Partition:** A partition $P$ of an interval $[a, b]$ is a set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ such that $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.

$\Delta x_i = x_i - x_{i-1}$

$||P||$: norm of partition $||P|| = \max(\Delta x_i)$

$x^*_{i}$: any (defined) number in $[x_{i-1}, x_i]$

A Riemann Sum $= \sum_{i=1}^{n} f(x^*_i) \Delta x_i$

The area under the graph of a positive function $f$ from $x = a$ to $x = b$ is

$$\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i) \Delta x_i$$
Warm-Up

Examples:

Given \( f(x) = x^2 + x \), write and compute a Riemann Sum to approximate the area under \( f \) from \( x = 0 \) to \( x = 3 \) using a partition \( P = \{0, 1.5, 2.5, 3\} \). Let \( x_i^* \) be the midpoint of each subinterval.

\[
\int_0^3 (x^2 + x) \, dx \approx \sum_{i=1}^{3} f(x_i^*) \Delta x_i
\]

\[
\begin{array}{c|c|c}
\text{base} & \text{height} \\
1.5 & (0.75) \\
1 & f(1.5) \\
0.5 & f(2.75) \\
\end{array}
\]

\[
(1.5)(2.35^2 + 6.35) + (1)(2^2 + 2)(0.5)(2.75^2 + 2.75)
\]
On Beyond Average: Find the exact area under the graph of \( f \) from \( x = 0 \) to \( x = 3 \). (HINT: use \( n \) equally spaced rectangles and take \( x^*_k \) = the right endpoint of each rectangle. Then let \( n \to \infty \))

\[
\int_0^3 (x^2 + x) \, dx = \lim_{n \to \infty} \frac{3}{n} \sum_{k=1}^{n} f\left( \frac{k}{n} \right)
\]

\[
= \lim_{n \to \infty} \frac{3}{n} \left( \frac{1}{2} \cdot \frac{9}{n^2} + \frac{3}{2} \right)
\]

\[
= \lim_{n \to \infty} \frac{3}{n} \left( \frac{9}{2n^2} + \frac{9}{2} \right)
\]

\[
= \lim_{n \to \infty} \frac{3}{n} \left( \frac{9}{2n^2} + \frac{9}{2} \right)
\]

\[
= \lim_{n \to \infty} \frac{3}{n} \left( \frac{9}{2n^2} + \frac{9}{2} \right)
\]

\[
= \lim_{n \to \infty} \frac{9}{2n^2} + \frac{9}{2} \cdot \frac{1}{n}
\]

\[
= \lim_{n \to \infty} \frac{9}{2n^2} + \frac{9}{2} \cdot \frac{1}{n}
\]

\[
= \lim_{n \to \infty} \frac{9}{2n^2} + \frac{9}{2n}
\]

\[
= 9 + \frac{9}{2} = \frac{27}{2}
\]