1 1.3: Vector Functions and Parametrized Curves

Definitions:
(Recall) function: a rule that assigns to each input a unique output

So far:
\[ f : \mathbb{R} \rightarrow \mathbb{R} \]
\[ x \mapsto y \]

Vector Valued function:

Now:
\[ F : \mathbb{R} \rightarrow \mathbb{R}^2 \]
\[ t \mapsto \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \]
\[ \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} \]
\[ \frac{dx}{dt} = (1+2t)^2 + t^2 \hat{j} \]
\[ \frac{dy}{dt} = 3t^2 \hat{j} \]

Parametrized Curve: the set of all points which correspond to an output
of the vector function

\[ \{(x,y) \mid \text{there is a } t \text{ with } \vec{r}(t) = (x,y) \hat{j} \} \]

Eliminating the Parameter

1) Solve one component for \( t \) and substitute into other component

\[ x = t^2 \]
\[ y = t^2 \]

2) Use identities to relate trig functions

\[ t = x-2 \rightarrow y = (x-2)^2 \]
Vector and Parametric Equations of a Line

Given:
1) a vector in the direction of \( \mathbf{v} \)
2) a vector corresponding to a point on line \( \mathbf{r}_0 \)

Vector Equation of Line

\[ \mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{v} \]

(Expand to get Parametric Equations)
Examples:

Given the curve parametrized by $r(t) = (t^2 + 1)i + (t^2 - 1)j$, determine when, if at all, the curve passes through the point $(3, 3)$.

\[
\begin{align*}
&\text{and} \\
&t^2 + 1 = 5 \\
&t^2 - 4 = 0 \\
&(t+2)(t-2) = 0 \\
&t = -2, 2
\end{align*}
\]

Yes, when $t = 2$ and $-2$. 

\[
\begin{align*}
&t^2 - 1 = 3 \\
&t^2 - 4 = 0 \\
&t = 2, -2
\end{align*}
\]
Describe the motion of a particle with position \( r(t) = \langle 2 \sin t, 3 \cos t \rangle \), \( 0 \leq t \leq 2\pi \).

1) \text{Start} \quad t = 0 \quad \Rightarrow \quad \hat{r}(0) = \langle 2 \sin 0, 3 \cos 0 \rangle = \langle 0, 3 \rangle \\
2) \text{End} \quad t = 2\pi \quad \Rightarrow \quad \hat{r}(2\pi) = \langle 2 \sin 2\pi, 3 \cos 2\pi \rangle = \langle 0, 3 \rangle \\
3) \text{Path (eliminate the parameter)} \\
\quad \frac{x}{2} = \sin t \quad \frac{y}{3} = \cos t \\
\quad \sin^2 t + \cos^2 t = 1 \\
\quad \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \\
4) \text{Direction} \quad t = \frac{\pi}{4} \quad \hat{r}(\frac{\pi}{4}) = \langle 2 \sin \frac{\pi}{4}, 3 \cos \frac{\pi}{4} \rangle = \langle 2, 0 \rangle \quad \text{Clockwise}
Find vector and parametric equations for the line passing through the points \((-3, 4)\) and \((2, 8)\).

Vector Equation:
\[
\vec{r}(t) = \vec{r}_0 + t \vec{v}
\]
\[
\vec{r}_0 = -3\hat{i} + 4\hat{j}
\]
\[
\vec{v} = (2\hat{i} + 8\hat{j}) - (-3\hat{i} + 4\hat{j})
\]
\[
\vec{v} = 5\hat{i} + 4\hat{j}
\]

Parametric Equation:
\[
\vec{r}(t) = (-3 + 5t)\hat{i} + (4 + 4t)\hat{j}
\]

or
\[
\begin{align*}
x &= -3 + 5t \\
y &= 4 + 4t
\end{align*}
\]
On Beyond Average:
Given $r(t) = \sin t \hat{i} + \cos^2 t \hat{j}$, eliminate the parameter to find the Cartesian equation of the curve. Is the point $(2, -3)$ on the curve? Sketch the graph.

$x = \sin t$
$y = \cos^2 t$

$s \in [0, 2\pi]

(2, -3) \text{ NOT on curve}

* Pay attention to range of $x$ and $y$!