Warm-Up

Assume that $x$ and $y$ are related by the equation $2x + xy^2 = 11$. Find $\frac{dy}{dx}$ at the point $(1, -3)$.

(a) $-\frac{11}{6}$
(b) $-\frac{3}{2}$
(c) $\frac{1}{3}$
(d) $\frac{11}{6}$
(c) $-\frac{1}{3}$
1 3.6: Implicit Differentiation

The equation $F(x, y) = 0$ implicitly defines a relation (not necessarily a function) between $y$ and $x$. The graph of $F(x, y) = 0$ is the set of all points $(x, y)$ such that the equation holds ($\{ (x, y) | F(x, y) = 0 \}$). Given a graph of an implicitly-defined relation, we can still talk about the slope of the line tangent to the curve at the given point.

**Method for Implicit Differentiation:**

1. Done when $y$ is not explicitly defined as a function of $x$.
2. Differentiate both sides of the equation, remembering that $y$ depends on $x$ (you can call it $y(x)$ if that helps)
3. Solve for $y'(x)$ or $\frac{dy}{dx}$. 
Examples:

Find \( \frac{dy}{dx} \) if \( y^5 + 3x^2y^2 + 5x^4 = 12 \).

Reminder that \( y = y(x) \)

\[
5(y(x))^4 \frac{dy}{dx} + 3x^2(2y(x) \frac{dy}{dx}) + (y(x))^2(6x) + 20x^2 = 0
\]

Replace \( y(x) \) with \( y \) again (and \( y'(x) \) with \( y' \))

\[
5y^4 \frac{dy}{dx} + 6x^2y \frac{dy}{dx} + 6xy^2 + 20x^3 = 0
\]

\[
5y^4 \frac{dx}{dy} + 6x^2y \frac{dx}{dy} = -6xy^2 - 20x^3
\]

\[
\frac{dy}{dx} = \frac{-6xy^2 - 20x^3}{5y^4 + 6x^2y}
\]
Implicit Differentiation

Problem Statement:
Assume \( x \) and \( y \) are related by \( 2x + xy^2 = 11 \). Find \( \frac{dy}{dx} \) at the point \( (x,y) = (1,-3) \).

Step 1 - Differentiate the equation with respect to \( x \).

Step 2 - Evaluate at the point.
That is, substitute \( (x,y) = (1,-3) \) into the previous equation.

Step 3 - Solve for the derivative.
\( \frac{dy}{dx} = \)

\[ \frac{dy}{dx} = \frac{1}{6} \]
Show that the curves $y = 3x^2$ and $x^2 + 2y^2 = 19$ are orthogonal.

\[ y' = 6x \]

**Correction:**

We are interested in where the curves intersect. (Solve the system)

\[
\begin{align*}
x^2 + 2(3x^2) &= 19 \\
18x^4 + x^2 - 19 &= 0 \\
(18x^2 + 19)(x^2 - 1) &= 0
\end{align*}
\]

or can use quadratic formula

\[
18x^2 + 19 = 0 \quad x^2 - 1 = 0
\]

No real soln. \( x = \pm 1 \)

\[
\begin{align*}
\text{If } x &= 1 \\
y &= 3
\end{align*}
\]

\[
\begin{align*}
\text{If } x &= -1 \\
y &= 3
\end{align*}
\]
The equations \( y = mx \) and \( x^2 + y^2 = r^2 \) represent families of curves for different constants \( r \) and \( m \). Show that these families of curves are orthogonal (for all values of the constants)

\[
y' = m \quad 2x + 2y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y} = \frac{-x}{mx} = \frac{-1}{m}
\]

\(:. \text{ curves are } \perp \text{ for all values of } m \text{ and } r\)

**NOTE**: We just proved that a line tangent to a circle is \( \perp \) to its diameter!
On Beyond Average: If \( \cos \left( \frac{x}{y} \right) = \sqrt{x^2 + y^2} \), find \( \frac{dy}{dx} \).

\[
\begin{align*}
-\sin \left( \frac{x}{y} \right) \left( \frac{y}{y^2} \right) & = \frac{1}{3} (x+y)^{\frac{2}{3}} \left( 1 + \frac{x^2}{y^2} \right) \\
-\sin \left( \frac{x}{y} \right) \left( \frac{1}{y} - \frac{x^2}{y^2} \right) & = \frac{1}{3} (x+y)^{\frac{2}{3}} \left( 1 + \frac{x^2}{y^2} \right)
\end{align*}
\]

\[
\begin{align*}
-\frac{1}{y} \sin \left( \frac{x}{y} \right) & + \frac{x}{y^2} \sin \left( \frac{x}{y} \right) \frac{dy}{dx} = \frac{1}{3} (x+y)^{\frac{2}{3}} + \frac{1}{3} (x+y)^{\frac{2}{3}} \frac{dy}{dx} \\
\frac{x}{y^2} \sin \left( \frac{x}{y} \right) \frac{dy}{dx} - \frac{1}{3} (x+y)^{\frac{2}{3}} \frac{dy}{dx} & = \frac{1}{3} (x+y)^{\frac{2}{3}} + \frac{1}{y} \sin \left( \frac{x}{y} \right)
\end{align*}
\]

\[
\frac{dy}{dx} \left( \frac{x}{y^2} \sin \left( \frac{x}{y} \right) - \frac{1}{3} (x+y)^{\frac{2}{3}} \right) = \frac{1}{3} (x+y)^{\frac{2}{3}} + \frac{1}{y} \sin \left( \frac{x}{y} \right)
\]

\[
\frac{dy}{dx} = \frac{\frac{1}{3} (x+y)^{\frac{2}{3}} + \frac{1}{y} \sin \left( \frac{x}{y} \right)}{\frac{x}{y^2} \sin \left( \frac{x}{y} \right) - \frac{1}{3} (x+y)^{\frac{2}{3}}}
\]