Warm-Up

If $f(x) = \frac{2 - x}{2 + x}$, which of the following is $f^{-1}$?

(a) $\frac{2 - 2x}{x + 1}$
(b) $\frac{x + 2}{x - 2}$
(c) $\frac{x + 2}{2 - x}$
(d) $\frac{2x - 2}{x + 1}$
(e) $\frac{2 - y}{y + 2}$
1 4.2: Inverse Functions

functions vs. one-to-one functions:

\[ \text{function: a rule that assigns a unique output to each input} \]
\[ \text{one-to-one function: a rule that assigns a unique output to a unique input} \]
\[ \text{(DEFN: only solution to } f(a) = f(b) \text{ is } a = b) \]

If \( f \) is one-to-one, the inverse of \( f \) is a function \( f^{-1} \) such that

\[
\begin{align*}
\text{if } y &= f'(x) \text{, then } \\
x &= f(y)
\end{align*}
\]

If \((a, b)\) is on the graph of \( y = f(x) \), then \((b, a)\) is on the graph of \( y = f'(x) \).

NOTE: graph of \( y = f'(x) \) is a reflection of \( y = f(x) \) about the line \( y = x \).

If \( f \) is one-to-one and differentiable at \( x = g(a) \), where \( g = f^{-1} \), then

\[
\begin{align*}
y &= f'(x) \quad \text{means} \\
x &= f(y) \quad \text{implicit diff} \\
1 &= f'(y) \\
\frac{dy}{dx} &= \frac{1}{f'(y)} \\
g'(x) &= \frac{1}{f'(g(a))} \quad \text{or} \quad g'(a) = \frac{1}{f'(g(a))}
\end{align*}
\]
Examples:

Show \( f(x) = \frac{2-x}{2+x} \) is one-to-one and find \( f^{-1} \):

\[
\begin{align*}
y &= \frac{2-x}{2+x} & \text{Switch } x \text{ and } y \\
x &= \frac{2-y}{2+y} & \text{Solve for } y \\
(2+y)x &= 2-y \\
2x + xy &= 2-y \\
x+y &= 2-2x \\
y(x+1) &= 2-2x \\
f^{-1}(x) &= y = \frac{2-2x}{x+1}
\end{align*}
\]
Given $g$ is the inverse of $f(x) = x^5 - x^3 + 4x$, find $g'(4)$.

\[ g'(4) = \frac{1}{f'(g(4))} \]

\[ f'(x) = 5x^4 - 3x^2 + 4 \]

\[ g'(4) = \frac{1}{f'(g(4))} \]

\[ y = g(4) \text{ means } \]

\[ 4 = f(y) \]

\[ 4 = y^5 - y^3 + 4y \]

\[ y = 1 \text{ by inspection (only solution since } f \text{ is one-to-one)} \]
DERRIVATIVES OF INVERSE FUNCTIONS

Maplet

\( f(x) = 2e^x \), \( y = e^y \), find \( g'(\frac{1}{e}) \)

\[ g'(\frac{1}{e}) = \frac{1}{f'(y(x))} \]

\[ f'(x) = 2e^x \]

\[ y = \frac{1}{e} \text{ means} \]

\[ \frac{1}{e} = f(y) \]

\[ \frac{1}{e} = 2e^y \]

\[ y = -1 \]

\[ g'(\frac{1}{e}) = \frac{1}{f'(-1)} \]

\[ = \frac{1}{2e^{-1}} \]

\[ = \frac{e}{2} \]
On Beyond Average: Find the inverse of \( f(x) = \sqrt{x-1} \).

\[
\begin{align*}
y &= \sqrt{x-1} \\
x &= \sqrt{y-1} \\
x^2 &= y-1 \\
f^{-1}(x) &= y = x^2 + 1
\end{align*}
\]

**NOT ONE-TO-ONE**

We can restrict the domain to make it one-to-one.

\[
f^{-1}(x) = x^2 + 1, \quad x > 0
\]

**NOTE:** \( g(x) = x^2 + 1, \quad x \leq 0 \) is the inverse of \( f(x) = \sqrt{x-1} \).