Warm-Up: Calculator Allowed!

Calculusium is a radioactive material with a half-life of 20 years. If you start out with 100 grams, how much is left (to the nearest tenth) after 40 years?

(a) 40.0 g
(b) 60.0 g
(c) 13.5 g
(d) 25.0 g
(e) 50.0 g
1 4.5: Exponential Growth and Decay

A solution to the differential equation \( y' = ky \) is: \[ y = Ce^{kt} \quad (C, k \text{ constants}) \]

Show: \[ y' = Ce^{kt} (k) \]

\[ Ce^{kt}(k) = k(Ce^{kt}) \]

\[ Cke^{kt} = Cke^{kt} \checkmark \]

(\text{in 308}) \( y = Ce^{kt} \) is the only solution to \( y' = ky \).

Exponential Growth and Decay:

Idea: \underline{The rate of change is a quantity is proportional to the amount present.}

Mathematically: \underline{Let } \( y \) \underline{quantity (amount)} \n
\[ y' = ky \]

So we know \[ y(t) = Ce^{kt} \]

\[ \frac{a}{b} = \text{some constant} \quad k \]

\[ a = kb \]
Goal: Use given information to find $C$ and $k$.

Examples:

RADIOACTIVE DECAY Maplet

\[ y' = ky, \quad y(0) = C \]

\[ y(t) = Ce^{kt} \]

When $t = 0$, $y = 100$.

\[ 100 = Ce^{0} \quad \Rightarrow \quad C = 100 \]

Use initial condition

\[ y(t) = 100e^{kt} \]

Use a subsequent data point ($y = 50$)

\[ 50 = \frac{100e^{kt}}{100} \]

\[ \ln \frac{1}{2} = 20k \]

\[ k = \frac{1}{2} \ln \frac{1}{2} \]

\[ y = 100e^{\frac{1}{2} \ln \frac{1}{2}} \]

\[ y = 100e^{-\frac{1}{2}} \]

\[ y = 100 \cdot \left(\frac{1}{2}\right)^{\frac{1}{2}} \]

\[ y = 100 \cdot \left(\frac{1}{2}\right)^{\frac{1}{2}} = 100 \left(\frac{1}{2}\right)^{\frac{1}{2}} = 25 \text{gms} \]
On Beyond Average: A 100-gallon drum contains 5 lbs of salt. Pure water enters the drum at a rate of 10 gallons per minute. The solution is thoroughly mixed and leaves the drum at the same rate. Find an equation which gives the amount of salt in the drum at any time $t$.

Let $y =$ amount of salt (lbs) 

$y' =$ rate in - rate out 

$\frac{dy}{dt} = 0 - \frac{10y}{100} (\text{lbs/min})$

$y' = -\frac{1}{10} y$

Solution to DE: 

$y = Ce^{-\frac{1}{10}t}$

$S = Ce^{-\frac{1}{10}(0)}$

$S = C$

$y(0) = 5 e^{-\frac{1}{10}t}$

NOTE: $\lim_{t \to \infty} y = 0$

when $t = 0$, $y = 5$
Newton’s Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference in temperature between the object and its surroundings. A metal ball is brought from 5°C weather into a 20°C room. One minute later, the ball has a temperature of 12°C. Write a function which gives the temperature of the ball at any time $t$.

Let $y =$ temperature of ball ($^\circ$C)
$t =$ time (min)

$y' = k(y - 20)$
stuff = $k(stuff)$
$y - 20 = C e^{kt}$

$y = 20 + C e^{kt}$
$5 = 20 + C e^{(0)}$
$5 = 20 + C$
$C = -15$
$y = 20 - 15 e^{kt}$
$k =$ subsequent data point when $t=1, y=12$

$12 = 20 - 15 e^{k(1)}$
$-8 = -15 e^k$
$\frac{-8}{-15} = e^k$

$\ln(\frac{-8}{-15}) = k$

$y = 20 - 15 e^{\ln(\frac{-8}{-15}) t}$

$\lim_{t \to \infty} y = 20$