10.6: Functions as Power Series

As stated previously, the power series \( \sum_{n=0}^{\infty} c_n(x - a)^n \) represents a function whose domain is the interval of convergence of the series. In this section, we use derivatives and integrals to obtain new power series.

Recall: \( \frac{d}{dx} (f(x) + g(x)) = f'(x) + g'(x) \)

\[ \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx \]

Claim: If \( \sum_{n=0}^{\infty} c_n(x - a)^n \) has radius of convergence \( r \), then \( \sum_{n=0}^{\infty} c_n n(x - a)^{n-1} \) has radius of convergence \( r \).

If \( \sum_{n=0}^{\infty} c_n(x - a)^n \) converges to \( f(x) \), what should \( \sum_{n=0}^{\infty} c_n n(x - a)^{n-1} \) converge to? \( f'(x) \)

In like manner, \( \int \left( \sum_{n=0}^{\infty} c_n(x - a)^n \right) \, dx \) has radius of convergence \( r \) and converges to \( \int f(x) \, dx \).
Examples: Given $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, find a power series for $f'(x)$ and $\int f(x) \, dx$. What is $f(x)$?

\[
f'(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}
\]

**NOTE:** if $m = n - 1$

\[
= \sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x)
\]

\[
\int f(x) \, dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)n!} = C + \sum_{m=1}^{\infty} \frac{x^m}{m!}
\]

**NOTE:** $m = n + 1$

\[
= C + \sum_{m=1}^{\infty} \frac{x^m}{m!} \quad \text{(take 1 from constant)}
\]

\[
\frac{1}{a^n}(e^x) = (e^x) ; \quad \int e^x \, dx = e^x + C
\]

\[
\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x
\]

*(Can show ROC = \infty)*
Determine a power series for \( f(x) = \arctan x \).

We know \( f'(x) = \frac{1}{1+x^2} \)

\[ f'(x) = \frac{1}{1-(-x^2)} \text{ looks like the Geometric Series } \]

\[ \frac{1}{1-r} = \sum_{n=0}^{\infty} (1)(-x^2)^{n-1} \text{ OR } \sum_{n=0}^{\infty} (1)(-x^2)^n \]

Integrate \[ \int \frac{1}{1+x^2} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} \text{ Properties of Exponents 4.1 } \]

\[ \tan^{-1} x = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \text{ Find C by letting } x = 0 \]

\[ 0 = \tan^{-1} 0 = C + \frac{0}{2n+1} \text{ C = 0} \]

\[ \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \]
Find the radius and interval of convergence of the series.

ROC: since we started with Geometric Series
Convergent when \(|r| < 1\)
\[ \left| 1 - x^2 \right| < 1 \]
\[ x^2 < 1 \quad \text{when} \quad -1 < x < 1 \]
\[ \text{ROC} = 1 \]

\[ x = -1 \quad \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} = -\frac{\pi}{4} \]
\[ x = 1 \quad \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \]

\[ \text{Interval: } [-1, 1] \]

Choosing an appropriate value of \(x\), how many terms of the series are needed to approximate \(\frac{\pi}{4}\) to within 0.001?

\[ \arctan(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{2n+1} \]
\[ \arctan x = \frac{\pi}{4} \]
\[ x = \pm \tan \frac{\pi}{4} \]

\[ \frac{\pi}{4} = \arctan(1) = \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} = \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} \quad \text{AH Series} \]
\[ |x - x_n| < a_{n+1} \]

We want \(a_{n+1} < 0.001\)

\[ \frac{1}{2n+3} \leq \frac{1}{1000} \]
\[ 1000 \leq 2n + 3 \]
\[ 997 \leq 2n \]
\[ N \geq \frac{997}{2} = 498.5 \quad \text{Need } 499 \text{ terms} \]
\[ f(x) = x \left( \frac{1}{(1+x)^2} \right) \]

Find a power series for \( f(x) = \frac{x}{(1+x)^2} \) and find its radius and interval of convergence.

\[ \sum \frac{1}{(1+x)^2} \, dx = -\frac{1}{1+x} = \frac{-1}{1-(-x)} \]

Geometric, \( a = -1, \ r = -x \)

\[ \frac{-1}{1+x} = \sum_{n=0}^{\infty} (-1)^n (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \]

Differentiate:

\[ \frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \]

\[ \frac{x}{(1+x)^2} = x \sum_{n=1}^{\infty} (-1)^n n x^{n-1} \]

\[ = \sum_{n=0}^{\infty} (-1)^{n+1} n x^n \]

**ROC:** Geometric, so need

\[ |r| = |-x| < 1 \]

\[ 1|x| < 1 \quad -1 < x < 1 \quad \text{ROC} = 1 \]

\[ x^2 \sum_{n=1}^{\infty} (-1)^{n+1} n (-1)^n = \sum_{n=1}^{\infty} (-1)^{2n+1} n = \sum_{n=0}^{\infty} -1 \cdot n \quad \text{divergent by Test for Divergence} \]

\[ x^2 \sum_{n=1}^{\infty} (-1)^{n+1} n (-1)^n = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n \quad \text{divergent by Test for Divergence} \]