1 13.4: Polar Coordinates

Let \((x, y)\) be a point in the (2-dimensional) plane. Define the polar coordinates of the point as follows:

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  r^2 &= x^2 + y^2 \\
  \tan \theta &= \frac{y}{x}
\end{align*}
\]
Graphs of Polar Equations:

\[ r = \# \quad \text{circle} \]
\[ \theta = \# \quad \text{line} \quad (m = \tan \#) \]

Other equations:

1) convert to Cartesian

\[ r = f(\theta) \]
2) analyze by “quarter-periods”

\[
\begin{array}{c|c|c}
\theta & r \\
\hline
[0, \frac{\pi}{2}] & ? \\
[\frac{\pi}{2}, \pi] & ? \\
\text{etc.} & \\
\end{array}
\]
Examples:

Plot the polar point \( \left( 2, \frac{4\pi}{3} \right) \) and find the Cartesian coordinates of the point.

\[
x = 2 \cos \frac{4\pi}{3} = -1
\]
\[
y = 2 \sin \frac{4\pi}{3} = -\sqrt{3}
\]

\( (-1, -\sqrt{3}) \)
Find the polar coordinates of the point \((5, -5\sqrt{3})\). (Make \(r > 0\) and \(0 \leq \theta \leq 2\pi\))

\[r^2 = 5^2 + (-5\sqrt{3})^2 = 25 + 75 = 100 \Rightarrow r = 10\]

\[\tan \theta = \frac{-5\sqrt{3}}{5} \Rightarrow \theta = \frac{\pi}{3}\] (ref. angle)

\[x > 0, y < 0 \Rightarrow \text{Quadrant IV}\]

Thus, \((10, \frac{5\pi}{3})\)
Sketch the region of the plane consisting of all points where $2 \leq r \leq 4$ and $\frac{3\pi}{4} \leq \theta \leq \pi$ and find the area of this region.

\[ A_{\text{outer slice}} = \pi (4)^2 \cdot \frac{\frac{4\pi}{3}}{2\pi} \]
\[ = 16\pi \cdot \frac{1}{8} = 2\pi \]

\[ A_{\text{inner slice}} = \pi (2)^2 \cdot \frac{\frac{4\pi}{3}}{2\pi} \]
\[ = 4\pi \cdot \frac{1}{8} = \frac{\pi}{2} \]

\[ A_{\text{region}} = 2\pi - \frac{\pi}{2} = \left[\frac{3\pi}{2}\right] \]

\[ \text{Note: Area of a sector} \]
\[ A = \pi r^2 \cdot \frac{\theta}{2\pi} = \frac{1}{2} r^2 \theta \]
Sketch the polar curve $r = \sin \theta$ and find a Cartesian equation for the curve.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\left[0, \frac{\pi}{2}\right]$</td>
<td>0..1</td>
</tr>
<tr>
<td>$\left[\frac{\pi}{2}, \pi\right]$</td>
<td>1..0</td>
</tr>
<tr>
<td>$\left[\pi, \frac{3\pi}{2}\right]$</td>
<td>0..-1</td>
</tr>
<tr>
<td>$\left[\frac{3\pi}{2}, 2\pi\right]$</td>
<td>-1..0</td>
</tr>
</tbody>
</table>

$r < 0$ so

$\text{Quad I}$

$\text{Quad II}$

Cartesian Equation

$r (r = \sin \theta)$

$r^2 = r \sin \theta$

$x^2 + y^2 = y$

$x^2 + y^2 - y + \frac{1}{4} = 0 + \frac{1}{4}$

$x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$

Circle

$r = \frac{1}{2}$

Center $(0, \frac{1}{2})$
Find a polar equation for the curve $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ and sketch the graph.

\[
2(r^2)^2 = 25(r^2 \cos^2 \theta - r^2 \sin^2 \theta) \\
2r^4 = 25r^2(\cos^2 \theta - \sin^2 \theta) \\
2r^4 = 25r^2 \cos 2\theta \\
2r^2 = 25 \cos 2\theta \\
\]

Period $= \frac{2\pi}{2} = \pi$

<table>
<thead>
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<th>$r$</th>
</tr>
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<tbody>
<tr>
<td>$[0, \frac{\pi}{4}]$</td>
<td>$\frac{5\sqrt{3}}{3}$...0 or $\frac{5\sqrt{3}}{3}$...0</td>
</tr>
<tr>
<td>$[\frac{\pi}{4}, \frac{3\pi}{4}]$</td>
<td>X X $r^2 &lt; 0$</td>
</tr>
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<td>0...$\frac{5\sqrt{3}}{3}$ or 0...$\frac{5\sqrt{3}}{3}$</td>
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**Polar Equation:**

\[
r = \frac{25}{2 \cos 2\theta}
\]