1 10.7: Taylor Series

Goal: Given any differentiable function \( f \), find a power series \( \sum_{n=0}^{\infty} c_n(x - a)^n \) which is equal to \( f(x) \) for all \( x \) on its interval of convergence.

Easier Look: Let \( a = 0 \) (This is also called a Maclaurin Series). We begin by assuming the first equation below is true for all \( x \) on the interval of convergence of the series:

\[
f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots
\]
If $T_N$ is the $N$th partial sum of the Taylor series of $f$ (also called the \textbf{nth-degree Taylor Polynomial} of $f$), and $R_N$ is the \textbf{remainder} (i.e., the sum of the rest of the terms), we can prove the series converges to $f(x)$ (for all $x$ in the interval of convergence) by proving $R_N(x) \to 0$.

\textbf{Example:} Find the Maclaurin Series for $f(x) = \sin x$ and explain why the series converges to $\sin x$. 
As we can see, it is easy to estimate $R_N(x)$ if the Taylor series is alternating (using the Alternating Series Estimation Theorem). If not, we can use the following:

**Taylor’s Inequality:** If $|f^{N+1}(x)| \leq M$ for $|x - c| \leq r$, then the remainder $R_N(x)$ of the $N$th degree Taylor polynomial of $f$ satisfies the inequality

**Example:** Find the Taylor series for $f(x) = e^x$ centered at $x = 2$. You do not have to prove that $R_N \rightarrow 0$. 

IMPORTANT MACLAURIN SERIES TO KNOW:

1. $e^x = $

2. $\frac{1}{1-x} = $

3. $\sin x = $

Use a Taylor Series to write $\int_{\pi}^{x} \frac{\sin t}{t} \, dt$ as a power series.
Some interesting results of Taylor Series: