1 7.5: Average Value of a Function

Goal: To find the average value of a function $f$ on a given interval $[a, b]$.

Average of $n$ values:

$$\frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Derivation of formula for average value of a function:

Let $P = \text{equally-spaced partition}$

avg of $f(x_i^*) = \frac{1}{n} \sum_{i=1}^{n} f(x_i^*)$

$$\Delta x = \frac{b-a}{n} \quad \frac{1}{h} = \frac{\Delta x}{b-a}$$

$$f_{\text{avg}} = \frac{\Delta x}{b-a} \sum_{i=1}^{n} f(x_i^*)$$

$$\lim_{|P| \to 0} f_{\text{avg}} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx$$
Geometric Interpretation:

\[ f_{\text{avg}} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \]

If \( f > 0 \):

\[(b-a) f_{\text{avg}} = \int_{a}^{b} f(x) \, dx \]

If \( f \) is cts, there is an \( x \)-value such that \( f(x) = f_{\text{avg}} \).
Examples: Find the average value of $f(x) = x^3$ from $x = 2$ to $x = 4$.

\[
\frac{f_{\text{avg}}}{a} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]
\[
= \frac{1}{4-2} \int_2^4 x^3 \, dx
\]
\[
= \frac{1}{2} \left( \frac{1}{4} x^4 \right)_2^4
\]
\[
= \frac{1}{2} (64 - 16) = 30
\]

\[
\text{Note} \quad f(x) = f_{\text{avg}}
\]
\[
x^3 = 30
\]
\[
x = \sqrt[3]{30} \quad 2 < x < 4
\]
Warm-Up (part b only)

The electric current in a household power supply is an alternating current modeled by \( i(t) = I \sin \omega t \).

a) Show that the average value of \( i \) over one period is 0.

b) The root mean square (rms) current is the square root of the average value of \( i^2 \) over one period. Calculate the rms current.

\[
\text{one period } = \left[ 0, \frac{2\pi}{\omega} \right]
\]

\[
i_{\text{avg}} = \frac{1}{\frac{2\pi}{\omega}} \int_0^{\frac{2\pi}{\omega}} I \sin(\omega t) \, dt
\]

\[
\begin{align*}
&= \frac{\omega}{2\pi} \left[ -\frac{I}{\omega} \cos(\omega t) \right]_0^{\frac{2\pi}{\omega}} \\
&= \frac{-I}{2\pi} \left( \cos \left( \frac{2\pi}{\omega} \right) - \cos(0) \right) \\
&= \boxed{0}
\end{align*}
\]
The electric current in a household power supply is an alternating current modeled by $i(t) = I \sin \omega t$.

a) Show that the average value of $i$ over one period is 0.

b) The root mean square (rms) current is the square root of the average value of $i^2$ over one period. Calculate the rms current.

\[
\begin{align*}
\text{b)} \quad i_{\text{rms}} &= \sqrt{\frac{1}{2\pi} \int_0^{2\pi} (I^2 \sin^2 \omega t) \, dt} \\
&= \sqrt{\frac{\omega I^2}{2\pi} \int_0^{2\pi} \sin^2 \omega t \, dt} \\
&= \sqrt{\frac{\omega I^2}{2\pi} \int_0^{2\pi} \frac{1}{2} (1 - \cos 2\omega t) \, dt} \\
&= \sqrt{\frac{\omega I^2}{4\pi} \left( t - \frac{1}{2\omega} \sin(2\omega t) \right)_0^{2\pi}} \\
&= \sqrt{\frac{\omega I^2}{4\pi} \left( \frac{2\pi}{\omega} - \frac{1}{2\omega} \sin(4\pi) \right) - 0 + 0} \\
&= \sqrt{\frac{\omega I^2}{4\pi} \cdot \frac{16\pi}{\omega}} \\
&= \sqrt{I^2} = \sqrt{\frac{I^2}{2}}
\end{align*}
\]
A tank is full of water. Find the work $W$ required to pump the water out of the spout. (Use 9.8 for $g$ and 3.14 for $\pi$. Round your answer to three significant digits.)

$$W = \quad \text{(in joules)}$$

$r = 4.5$
$h = 1.5$

$$F = \rho g \pi r^2 h$$
$$r = x$$
$$h = dy$$

$$W_{\text{sub}} = (\rho g \pi x^2 \ dy)(6-y)$$

$$W_{\text{tor}} = \int_{4.5}^{4.5} \rho g \pi (4.5^2-y^2)(6-y) \ dy$$
A heavy rope, 60 ft long, weighs 0.7 lb/ft and hangs over the edge of a building 160 ft high.

(a) How much work $W$ is done in pulling the rope to the top of the building?

$$W = \boxed{\text{ft-lb}}$$

(b) How much work $W$ is done in pulling half the rope to the top of the building?

$$W = \boxed{\text{ft-lb}}$$

\[\begin{align*}
\text{a) Partition method (like tank problems)} & \quad W_{tot} = \int_0^{60} 0.7y \, dy \\
\text{b) Force as function of distance} & \quad W = \int_0^{30} (42 - 0.7x) \, dx
\end{align*}\]