1 10.1: Sequences

Definitions:

sequence \( \{a_n\} \):

\[
\lim_{n \to \infty} a_n = L
\]

Sequences defined using real-valued functions:

Limit Laws: Given \( a_n \) and \( b_n \) converge and \( c \) is a constant, then:

1. \[
\lim_{n \to \infty} (a_n + b_n) =
\]
2. \[
\lim_{n \to \infty} (a_n - b_n) =
\]
3. \[
\lim_{n \to \infty} ca_n =
\]
4. \[
\lim_{n \to \infty} a_nb_n =
\]
5. \[
\lim_{n \to \infty} \frac{a_n}{b_n} =
\]
6. If \( \lim_{n \to \infty} |a_n| = 0 \), then
7. If \( f \) is a continuous function and \( a_n \to L \), then
8. **Squeeze Theorem**:

Examples: Find the limits of the following sequences:

a) \( a_n = \frac{\ln(n + e^{3n})}{n} \)
b) $a_n = \arctan \left( \frac{n}{n+1} \right)$

c) $a_n = \frac{(-1)^{n+1}}{2n+1}$

More Definitions:

Monotonic sequence:

$\{a_n\}$ is increasing for $n \geq N$ if and only if

implies: (1) $a_{n+1} - a_n > 0$;
(2) $\frac{a_{n+1}}{a_n} > 1$ if $a_n > 0$;
(3) If $a_n = f(n)$ for some real-valued function $f$, then $f' > 0$.

$\{a_n\}$ is decreasing for $n \geq N$ if and only if

implies:

$\{a_n\}$ is bounded above (below) if and only if

Monotone Sequence Theorem:
On Beyond Average:

Find the limit of \( a_n = (\sqrt{n + 1} - \sqrt{n})\sqrt{n + \frac{1}{2}} \)

Given \( a_n = \frac{1000^n}{n!} \), show \( a_n \) is decreasing (for \( n > \)some \( N \)) and bounded below. What is the limit of this sequence, and why?
Proof by Induction (Appendix E)

Example:

Given $a_0 = 1$ and $a_{n+1} = \frac{1}{3}a_n + 1$:

a) Show $a_n$ is increasing and $0 < a_n < 2$ for all $n$.

b) Find $\lim_{n \to \infty} a_n$. 