1 7.3: Volume by Cylindrical Shells

The volume of a cylindrical shell is given by:

\[ V_{shell} = 2\pi r h \, dr \]

Key: Look at a single shell and determine the radius, height, and thickness.
**Examples:** Find the volume of the solid formed by rotating the region bounded by the curves \( y = \sqrt{x}, \ x = 0, \) and \( y = 2 \) about the \( y \)-axis.

\[
V = \int_{0}^{4} 2\pi x (2 - \sqrt{x}) \, dx \\
= 2\pi \int_{0}^{4} (2x - x^{\frac{3}{2}}) \, dx \\
= \frac{32\pi}{3}
\]
Shells vs. Slices

(See drawings in lecture). Notice that when using slices to set up our integrals, we are integrating with respect to the “height” variable (perpendicular to the radius). However, when using shells to set up our integral, we are integrating with respect to the “thickness” variable (in the same direction as the radius). This can allow us to decide which method to use, as summarized in the table below:

<table>
<thead>
<tr>
<th>f(x)</th>
<th>Rotation about a horizontal axis</th>
<th>Rotation about a vertical axis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int 2\pi f(x) dx$</td>
<td>Slices</td>
<td>$\int 2\pi x (x-a) dx$</td>
</tr>
<tr>
<td>$\int 2\pi x (x-a) dx$</td>
<td>Shells</td>
<td>$\int 2\pi f(x) dx$</td>
</tr>
</tbody>
</table>

$\text{Same } \Rightarrow \text{ Slices}$

$\text{Different } \Rightarrow \text{ Shells}$
Examples: Set up, but do not compute, an integral to find the volume of the solid obtained by rotating the region bounded by \( x = 4 - y^2 \) and \( x = 8 - 2y^2 \) about the \( x \)-axis.

\[
V = \int_0^2 2\pi y (4-y^2) \, dy
\]

Axis: \( x \)  \quad f(y) \rightarrow \text{Shells}

\[
V_{\text{shell}} = 2\pi r \, h \, dr
r = y - 0 = y
\, dr = dy
h = R - L = (8 - 2y^2) - (4 + y^2) = 4 - y^2
\]

Boundaries:
- \( 4 - y^2 = 8 - 2y^2 \)
- \( y^2 - 4 \geq 0 \)
- \( y = \pm 2 \), top half only
Set up, but do not compute, an integral to find the volume of the solid formed by rotating the region bounded by the curves $y = x^2 + 1$ and $y = 3 - x$ about the $x$-axis.

\[ V = \pi \int_{-x^2+1}^{3-x} \left[ (3-x)^2 - (x^2+1)^2 \right] dx \]

**Boundaries**
- $x^2+1 = 3 - x$
- $x^2 + x - 2 = 0$
- $(x+2)(x-1)=0$
- $x=-2 \quad x=1$

$x-axis, \quad f(x) \rightarrow Slices (washers)$

\[ V = \pi r_{out}^2 h - \pi r_{in}^2 h \]

\[ h = dx \]

\[ r_{out} = (3-x) - 0 \]

\[ r_{in} = (x^2+1) - 0 \]
On Beyond Average:

Find the volume of the solid formed by rotating the region bounded by the curves $y = \sqrt{x}$, $x = 0$, and $y = 2$ about the line $y = -1$.

**NOTE:** Slices:

$$V = \pi \int_0^4 \left(3^2 - (x^2 + 1)^3\right) \, dx$$

$$x = y^2$$

$\text{Shell Method}$

$$V = 2\pi \int_0^2 y \cdot dy$$

$$r = y - (-1) = y + 1$$

$$dr = dy$$

$$h = R - L = y^2 - 0$$

$$V = \frac{2\pi}{5} \left(\frac{y^3}{3} + y^2\right) \, dy$$

$$= \frac{2\pi}{5} \left(4 + \frac{8}{5}\right)$$
Find the volume of the spherical cap formed by rotating the region bounded by \( x^2 + y^2 = 2 \) and \( y = 1 \) about the \( y \)-axis.

\[
\begin{align*}
V &= \int_{-1}^{1} \pi \left( \sqrt{2-y^2} \right)^2 \, dy \\
&= \pi \int_{-1}^{1} (2-y^2) \, dy \\
&= 2\pi \int_{0}^{1} x \left( \sqrt{2-x^2} - 1 \right) \, dx \\
&= 2\pi \left[ \frac{x}{2} \right]_{0}^{1} - 2\pi \int_{0}^{1} x \, dx \\
&= \pi - \pi \left[ \frac{x^2}{2} \right]_{0}^{1} \\
&= \pi - \frac{\pi}{2} = \frac{\pi}{2} 
\end{align*}
\]