1 10.1: Sequences

Definitions:

sequence \( \{a_n\} \):

\[
\lim_{n \to \infty} a_n = L
\]

Sequences defined using real-valued functions:

Limit Laws: Given \( a_n \) and \( b_n \) converge and \( c \) is a constant, then:

1. \( \lim_{n \to \infty} (a_n + b_n) = \)
2. \( \lim_{n \to \infty} (a_n - b_n) = \)
3. \( \lim_{n \to \infty} ca_n = \)
4. \( \lim_{n \to \infty} a_nb_n = \)
5. \( \lim_{n \to \infty} \frac{a_n}{b_n} = \)
6. If \( \lim_{n \to \infty} |a_n| = 0 \), then
7. If \( f \) is a continuous function and \( a_n \to L \), then
8. **Squeeze Theorem**:

Examples: Find the limits of the following sequences:

a) \( a_n = \frac{\ln(5 + e^n)}{5n} \)
b) \( a_n = \arctan \left( \frac{n}{n+1} \right) \)

c) \( a_n = \frac{(-1)^{n+1}}{2n+1} \)

**More Definitions:**

Monotonic sequence:

\( \{a_n\} \) is increasing for \( n \geq N \) if and only if

implies:

1. \( a_{n+1} - a_n > 0 \);
2. \( \frac{a_{n+1}}{a_n} > 1 \) if \( a_n > 0 \);
3. If \( a_n = f(n) \) for some real-valued function \( f \), then \( f' > 0 \).

\( \{a_n\} \) is decreasing for \( n \geq N \) if and only if

implies:

\( \{a_n\} \) is bounded above (below) if and only if

**Monotone Sequence Theorem:**
On Beyond Average:

Find the limit of \( a_n = (\sqrt{n+1} - \sqrt{n})\sqrt[n]{n + \frac{1}{2}} \)

Given \( a_n = \frac{1000^n}{n!} \), show \( a_n \) is decreasing (for \( n > \text{some } N \)) and bounded below. What is the limit of this sequence, and why?
Example:

Given \( a_0 = 1 \) and \( a_{n+1} = \frac{1}{3}a_n + 1 \):

a) Show \( a_n \) is increasing and \( 0 < a_n < 2 \) for all \( n \).

b) Find \( \lim_{n \to \infty} a_n \).