Warm-Up

Let $a_n = \frac{n^2 + 1}{2^n}$. Which statement about the series $\sum_{n=0}^{\infty} a_n$ is true?

(a) The series converges because $a_n \to 0$.

(b) The series converges because $\frac{a_{n+1}}{a_n} \to \frac{1}{2}$.

(c) The series diverges because $a_n \not\to 0$.

(d) The series diverges because $a_n > \frac{1}{2^n}$.

(e) None of the above statements is true.
10.4: Other Convergence Tests

(Series Convergence Tests continued—what if terms not all positive?)

VI. The Alternating Series Test: If \( \{b_n\} \) is positive, decreasing, and \( b_n \to 0 \), then the series \( \sum_{n=1}^{\infty} (-1)^{n+1} b_n \) converges.

Illustration: Showing that the Alternating Harmonic Series \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \) is convergent:

\[
= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots
\]

\[
|\sum_{n=1}^{N} (-1)^{n} b_n| \leq \frac{1}{N+1}
\]

\[
\lim_{N \to \infty} \frac{1}{N+1} = 0
\]

\[
\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges by AST}
\]

NOTE that \( \sum_{n=1}^{\infty} (-1)^{n} b_n \) is convergent (to a sum \( s \)) by the Alternating Series Test then

\[
|s - S_N| \leq b_{N+1}
\]
Definitions:

Absolutely Convergent Series:

A series $\sum a_n$ is absolutely convergent if and only if $\sum |a_n|$ converges.

If $a_n = \left(\frac{\sin n}{n}\right)$, $|a_n| = \frac{1}{n}$, $\sum |a_n|$ diverges so the Alternating Harmonic Series converges but NOT absolutely.

Conditionally Convergent Series (NOT in text!):

$$\sum \frac{(-1)^n}{n^2}$$

$|a_n| = \frac{1}{n^2}$ pos, dec, $\to 0$ so series converges by AST.

But $\sum |a_n| = \sum \frac{1}{n^2}$ converges by $p$-Test so

Theorem: If $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.

$\sum \frac{|a_n|}{n^2}$ converges absolutely.
VII. The Ratio Test:

If \[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \] then:

1) if \( L < 1 \), the series \( \sum a_n \) converges absolutely.
2) if \( L > 1 \), the series \( \sum a_n \) diverges.
3) if \( L = 1 \), ?

\[ \sum_{n=1}^{\infty} \frac{n^2 + 1}{2^n} \]

\[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2 + 1}{2^{n+1}} \cdot \frac{2^n}{n^2 + 1} \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n^2 + 2n + 2}{2n^2 + 2} = \frac{1}{2} < 1 \]

.: the series converges absolutely by Ratio Test.
Guidelines to Applying Convergence Tests:

<table>
<thead>
<tr>
<th>If $a_n$</th>
<th>Then try using this test</th>
</tr>
</thead>
<tbody>
<tr>
<td>does not $\to 0$</td>
<td>Test for Divergence</td>
</tr>
<tr>
<td>has factorials and/or exponentials</td>
<td>Ratio Test</td>
</tr>
<tr>
<td>alternates signs $(-1)^n$ or $(-1)^{n+1}$</td>
<td>Alternating Series Test</td>
</tr>
<tr>
<td>involves fractions with powers of $n$</td>
<td>Comparison or Limit Test</td>
</tr>
<tr>
<td>comes from a decreasing, easy-to-integrate function</td>
<td>Integral Test</td>
</tr>
</tbody>
</table>

*May have to test for absolute convergence by testing $|a_n|$. "
\[ \sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^2} \]

Not Alternating, use Integral Test estimator

For the last series, \( s_{100} = 1.037392940198405 \). Estimate how close this is to the actual sum of the series.

\[
S - S_{100} \leq \int_{100}^{\infty} \frac{\sin\left(\frac{1}{x}\right)}{x^2} \, dx \\
= \lim_{N \to \infty} \int_{100}^{N} \frac{\sin\left(\frac{1}{x}\right)}{x^2} \, dx \\
= \lim_{N \to \infty} \left[ \cos\left(\frac{1}{x}\right) \right]_{100}^{N} \\
= \lim_{N \to \infty} \left[ \cos\left(\frac{1}{N}\right) - \cos\left(\frac{1}{100}\right) \right] \\
= \left( 1 - \cos\left(\frac{1}{100}\right) \right) \approx 5 \times 10^{-5} \text{ on MATLAB}
\]

Estimate sum to 3 decimal places (error < .001 or .0005)

\[
S - S_N < \int_{N}^{\infty} \frac{\sin\left(\frac{1}{x}\right)}{x^2} \, dx < 0.0005
\]

Solve for \( N \)

\[
1 - \cos\left(\frac{1}{N}\right) < 0.0005 \\
0.9995 < \cos\left(\frac{1}{N}\right)
\]

\[
31.621 = \frac{1}{\cos^2(0.9995)} < N
\]

need 32 terms
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3}
\]

\[a_n = \frac{1}{n+3}\]

How many terms are needed to find the sum of the third and fourth series correct to within 0.001?

\[|S - S_n| \leq a_{n+1} < 0.001\]

\[
\frac{1}{N+4} < 0.001 \quad \text{Solve for } N
\]

\[
\frac{1}{10000} < 0.001
\]

\[10000 < N+4\]

\[N > 9996 \quad \text{and } 9997 \text{ terms}\]