10.6: Functions as Power Series

As stated previously, the power series \( \sum_{n=0}^{\infty} c_n(x - a)^n \) represents a function whose domain is the interval of convergence of the series. In this section, we use derivatives and integrals to obtain new power series.

Recall: \( \frac{d}{dx} (f(x) + g(x)) = \int (f(x) + g(x)) \, dx = \)

Claim: If \( \sum_{n=0}^{\infty} c_n(x - a)^n \) has radius of convergence \( r \), then \( \sum_{n=0}^{\infty} c_n n(x - a)^{n-1} \) has radius of convergence \( r \).

If \( \sum_{n=0}^{\infty} c_n(x - a)^n \) converges to \( f(x) \), what should \( \sum_{n=0}^{\infty} c_n n(x - a)^{n-1} \) converge to?

In like manner, \( \int \left( \sum_{n=0}^{\infty} c_n(x - a)^n \right) \, dx \) has radius of convergence \( r \) and converges to

Examples: Given \( f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \), find a power series for \( f'(x) \) and \( \int f(x) \, dx \). What is \( f(x) \)?
Find a power series for $f(x) = \frac{1}{1 + x^2}$ and use it to determine a power series for $f(x) = \arctan x$.

Find the radius and interval of convergence of the series.

Choosing an appropriate value of $x$, how many terms of the series are needed to approximate $\frac{\pi}{4}$ to within 0.001?
On Beyond Average:

Find the sum of $\sum_{n=0}^{\infty} nx^n$.

If $g(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \cdots + \frac{(-1)^n}{n}x^n + \cdots$, find $g\left(\frac{1}{2}\right)$.