10.9: Error Analysis in Taylor Polynomials

**Recall:** We can find the Taylor series of any differentiable function \( f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \) where \( c_n = \)

However, this is not very practical. It is true, however, that we can approximate the function with a finite polynomial by looking at partial sums.

**Recall:** The \( N \)th degree Taylor polynomial of \( f \) at \( x = a \):

The **Remainder** of the \( N \)th degree Taylor polynomial is given by
\( R_N(x) = \)

The question, then, is how large a polynomial is necessary to achieve a desired accuracy for the function on a given interval? That is, how far off are we at most at any point on a given interval when we stop the series at a given value of \( N \)?

Recall that if the series is an Alternating Series, then \( |R_N(x)| \leq \)

Also recall Taylor’s Inequality:

Graphical analysis of the error is another method which will be done in Matlab.
Examples:

Use a 3rd degree Taylor polynomial at $a = 0$ to approximate $e^x$ on the interval $[-1, 1]$ (which can then be used to approximate $e$) and determine the accuracy of your results using the remainder theorem.

Using the fact that $\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \cdots$, find the 4th degree Taylor polynomial approximation at $a = 0$ for $\ln(1 + x^2)$. Using this, estimate $\int_0^{1/2} \ln(1 + x^2) \, dx$. Estimate the error in using this approximation.
On Beyond Average:

Determine the degree of the Taylor Polynomial needed to approximate \[ \int_0^{0.1} \sin(x^2) \, dx \] to within $10^{-10}$ accuracy. (Calc required)

In Einstein’s special theory of relativity, the relativistic generalization of the kinetic energy of an object is given by

\[
K = mc^2 \left( \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right)
\]

Here \( m \) is the object’s mass, \( c \) is the speed of light, and \( v \) is the speed of the object. Show that, for everyday speeds (i.e., whenever \( v \) is VERY MUCH LESS than \( c \)), the above expression reduces to the classical kinetic energy of Newtonian theory, \( K = \frac{1}{2} mv^2 \): 

(a) Compute the first 3 terms of the Maclaurin series for \( f(x) = (1 + x)^{-1/2} \)

(b) Substitute \( x = -\frac{v^2}{c^2} \) into (a) to get an approximate series for \( \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} \)

(c) Substitute (b) into the original expression