9. Let \( f(x) \) be the function whose Taylor series at 3 is
\[
f(x) = \sum_{n=1}^{\infty} \frac{1 + \sqrt{n}}{(n+1)!} (x-3)^n.
\]
Find \( f^{(16)}(3) \), i.e., find the 16-th derivative of \( f \) at 3.

(a) \( \frac{5}{17!} \)  
(b) \( \frac{5}{16!} \)  
(c) \( \frac{5}{17} \)  
(d) 5  
(e) \( 5(3)^{16} \)
10.9: Error Analysis in Taylor Polynomials

**Recall:** We can find the Taylor series of any differentiable function \( f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \) where

\[
    c_n = \frac{f^{(n)}(a)}{n!}
\]

\[
    f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

However, this is not very practical. It is true, however, that we can approximate the function with a finite polynomial by looking at partial sums.

**Recall:** The \( N \)th degree Taylor polynomial of \( f \) at \( x = a \):

\[
    T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

\[
    f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots
\]

**Quadratic Approximation**

**Linear Approx**

The **Remainder** of the \( N \)th degree Taylor polynomial is given by

\[
    R_N(x) = \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
\]

\[
    = f(x) - T_N(x)
\]

\[
    S - S_N
\]
The question, then, is how large a polynomial is necessary to achieve a desired accuracy for the function on a given interval? That is, how far off are we at most at any point on a given interval when we stop the series at a given value of $N$?

Recall that if the series is an Alternating Series, then $|R_N(x)| \leq b_{N+1} = \frac{f^{N+1}(a)}{(N+1)!} |x-a|^{N+1}$

Also recall Taylor's Inequality: if $|f^{N+1}(x)| \leq M$ on an interval, then $|R_N(x)| \leq \frac{M}{(N+1)!} |x-a|^{N+1}$ (Given on exam)

Graphical analysis of the error is another method which will be done in Matlab.
Examples:

Use a 3rd degree Taylor polynomial at \( a = 0 \) to approximate \( e^x \) on the interval \([-1, 1]\) (which can then be used to approximate \( e \)) and determine the accuracy of your results using the remainder theorem.

We know

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

\[
T_3(x) = \frac{x^3}{3!} + \frac{x^2}{2} + x + 1
\]

\[
|R_n(x)| \leq \frac{M}{(N+1)!} |x-a|^{N+1}
\]

\[N = 3, a = 0\]

\[M \text{ upper bound on } 4^{th} \text{ deriv}\]

\[
|R_n(x)| \leq \frac{3}{4!} |x|^4
\]

\[\text{on } [-1,1]\]

\[\leq \frac{3}{4!} (1)^4 = \frac{1}{8}\]

\[M = e < 3\]

\[\text{Graph of } |R_3(x)|\]

\[
|R_3(x)| \leq 0.06 < \frac{1}{8}
\]
Using the fact that $\ln(1+t) \approx 1 - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \cdots$, find the 4th degree Taylor polynomial approximation at $a = 0$ for $\ln(1 + x^2)$. Using this, estimate $\int_0^{1/2} \ln(1 + x^2) \, dx$. Estimate the error in using this approximation.

\[
\ln(1 + t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} t^n
\]

\[
\ln(1 + x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{2n}
\]

\[
T_4(x) = \sum_{n=0}^{3} \frac{(-1)^{n-1}}{n} x^{2n} = \boxed{x^2 - \frac{1}{2}x^4}
\]

\[
\int_0^{1/2} \ln(1 + x^2) \, dx \approx \int_0^{1/2} \left( x^2 - \frac{1}{2}x^4 \right) \, dx
\]

\[
= \left[ \frac{1}{3} x^3 - \frac{1}{10} x^5 \right]_0^{1/2} = \frac{1}{48} - \frac{1}{320}
\]

**Alt Series**

\[
| r_4(x) | = | f(x) - T_4(x) | \leq b_3 = \frac{1}{3} |x|^{6} \quad \text{on} \quad [0, \frac{1}{2}]
\]

\[
S - S_2 \leq \frac{1}{3} \left( \frac{1}{2} \right)^6
\]
On Beyond Average:

Determine the degree of the Taylor Polynomial needed to approximate \( \int_0^{0.1} \sin(x^2) \, dx \) to within \( 10^{-10} \) accuracy. (Calc required)

\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]

Replace \( x \) with \( x^2 \) (NOT squaring!)

\[
\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(4n+2)!}
\]

\[
\int_0^{0.1} \sin(x^2) \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n (0.1)^{4n+2}}{(4n+2)(4n+3)!}
\]

Alternative Series

\[
|S - S_N| \leq b_{n+1} < 10^{-10}
\]

On Matlab:

\[
N \geq 1
\]

\[
\int_0^{0.1} \sin(x^2) \, dx \approx \frac{(0.1)^3}{3(1)!} - \frac{(0.1)^7}{7(3)!}
\]
In Einstein's special theory of relativity, the relativistic generalization of the kinetic energy of an object is given by

\[ K = mc^2 \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \]

Here \( m \) is the object's mass, \( c \) is the speed of light, and \( v \) is the speed of the object. Show that, for everyday speeds (i.e., whenever \( v \) is VERY MUCH LESS than \( c \)), the above expression reduces to the classical kinetic energy of Newtonian theory, \( K = \frac{1}{2}mv^2 \):

(a) Compute the first 3 terms of the Maclaurin series for \( f(x) = (1 + x)^{-1/2} \)

(b) Substitute \( x = -\frac{v^2}{c^2} \) into (a) to get an approximate series for \( \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} \)

(c) Substitute (b) into the original expression

\[
\begin{align*}
\text{a)} & \quad f(x) = (1+x)^{-1/2} \quad f(0) = 1 \quad T_2(x) = 1 - \frac{1}{2}x + \frac{3}{8}x^2 \\
& \quad f'(x) = -\frac{1}{2}(1+x)^{-3/2} \quad f'(0) = \frac{1}{2} \\
& \quad f''(x) = \frac{3}{4}(1+x)^{-5/2} \quad f''(0) = \frac{3}{4} \\
\text{b)} & \quad T_2\left(-\frac{v^2}{c^2}\right) = 1 - \frac{1}{2}\left(-\frac{v^2}{c^2}\right) + \frac{3}{8}\left(-\frac{v^2}{c^2}\right)^2 \\
& \quad = 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} \\
\text{c)} & \quad K = mc^2 \left( 1 - \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \right) - 1 \\
& \quad = mc^2 \left( \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} \right) \\
& \quad = m \left( \frac{v^2}{2} + \frac{3v^4}{8c^4} \right) \\
& \quad \text{so since } v \ll c, \quad K = \frac{1}{2}mv^2
\end{align*}
\]