1 7.2: Volume by Slicing

The volume of a prism with a base of area $B$ and a height $h$ is given by $V = Bh$.

If the height is not constant, we can, in many cases, “slice” the solid thin enough so that the height is constant.

**Example**: Find the volume of a square pyramid whose height is $h$ and whose base is $s$ by $s$.

\[
\begin{align*}
\frac{x}{s} &= \frac{h-y}{h} \\
hx &= sh - sy \\
x &= s - \frac{s}{h}y
\end{align*}
\]

\[
V_{\text{slice}} = (x_i^2) \Delta y_i
\]

\[
V_{\text{tot}} = \int_{0}^{h} \sum_{i=1}^{n} (x_i^2) \, dy_i
\]

\[
\frac{x}{s} = \frac{h-y}{h}
\]

\[
x = s - \frac{s}{h}y
\]

\[
x^2 = \frac{s^2}{h^2} y^2
\]

\[
hx = sh - sy
\]

\[
\int_{0}^{h} (s - \frac{s}{h}y)^2 \, dy
\]

\[
= \int_{0}^{h} \left( \frac{2s^2}{h^2} y + \frac{s^2}{h^3} y^2 \right) \, dy
\]

\[
= s^2 y - \frac{2s^2}{h} \left( \frac{1}{2} y^2 \right) + \frac{s^2}{h^2} \left( \frac{1}{3} y^3 \right) \bigg|_{0}^{h}
\]

\[
= s^2 h - \frac{s^2}{h} \left( \frac{1}{2} h^2 \right) + \frac{s^2}{h^2} \left( \frac{1}{3} h^3 \right)
\]

\[
= \frac{1}{3} s^2 h
\]
If the solid is generated by rotating a region about an axis, then the bases of the slices will be circles and the area will be \( B = \pi r^2 \). If there is a hole in the solid, we can find the volume by finding the volume of the outer solid (without the hole) minus the volume of the hollowed out portion.

**Examples:**

Find the volume of the solid formed by rotating the region bounded by \( y = 2 \) and \( y = 1 + x^2 \) about the \( x \)-axis.

\[
V = \int_0^1 \pi (2^2) \, dx - \pi \left(1 + x^2\right)^2 \, dx
\]

\[
= \int_0^1 \pi \left(2^2 - (1 + x^2)^2\right) \, dx
\]

\[
= \pi \int_0^1 \left(4 - 1 - 2x^2 - x^4\right) \, dx
\]

\[
= \pi \left[4x - \frac{2}{3}x^3 - \frac{1}{5}x^5\right]_0^1
\]

\[
= 2\pi \left(3 - \frac{2}{3} - \frac{1}{5}\right) = \frac{64\pi}{15}
\]
Find the volume of the solid formed by rotating the region bounded by $x = \cos y$, $x = 0$, $y = 0$, and $y = \frac{\pi}{2}$ about the $y$-axis.

\[ V = \int_{0}^{\frac{\pi}{2}} \pi \cos^2 y \, dy \]
\[ = \int_{0}^{\frac{\pi}{2}} \frac{\pi}{2} (1 + \cos 2y) \, dy \]
\[ = \frac{\pi}{2} \left[ y + \frac{1}{2} \sin 2y \right]_{0}^{\frac{\pi}{2}} \]
\[ = \frac{\pi}{2} \left( \frac{\pi}{2} + \frac{1}{2} \sin \pi - 0 - \frac{1}{2} \sin 0 \right) \]
\[ = \frac{\pi^2}{4} \]
Find the volume of the solid formed by rotating the region bounded by the curves \( y = \sqrt{x}, \ x = 0, \) and \( y = 2 \) about the \( y \)-axis.

\[
V = \int_{0}^{2} \pi (y^2) \, dy = \pi \left[ \frac{1}{3} y^3 \right]_{0}^{2} = \frac{32 \pi}{3}
\]

\[
x = y^2
\]

\[
y = \sqrt{x}
\]

\[
y = 2
\]
On Beyond Average:

Set up, but do not evaluate, an integral to find the volume of the solid formed by rotating the region in the previous example about the line $y = -1$. 

\[ V = \int_0^1 \pi \left( 3^2 - (\sqrt{x} + 1)^2 \right) \, dx \]
Find the volume of the solid whose base is the triangle with corners (0, 0), (0, 1) and (1, 0) and whose cross sections perpendicular to the x-axis are semicircles.

\[
V = \frac{1}{2} \pi \int_{0}^{1} \left( \frac{1}{4} (1-x)^2 \right) \, dx
\]

\[
= \frac{\pi}{2} \int_{0}^{1} \frac{1}{4} (1-x)^2 \, dx
\]

\[
= \frac{\pi}{8} \left( \frac{1}{3} (1-x^3) \right) \bigg|_{0}^{1}
\]

\[
= \frac{\pi}{8} (0 - 0)
\]

\[
= \frac{\pi}{24}
\]