1 10.1: Sequences

Definitions:

sequence \( \{a_n\} \):

\[
\lim_{n \to \infty} a_n = L
\]

Sequences defined using real-valued functions:

Limit Laws: Given \( a_n \) and \( b_n \) converge and \( c \) is a constant, then:

1. \[
\lim_{n \to \infty} (a_n + b_n) = 
\]
2. \[
\lim_{n \to \infty} (a_n - b_n) = 
\]
3. \[
\lim_{n \to \infty} ca_n = 
\]
4. \[
\lim_{n \to \infty} a_nb_n = 
\]
5. \[
\lim_{n \to \infty} \frac{a_n}{b_n} = 
\]
6. If \[
\lim_{n \to \infty} |a_n| = 0, \text{ then}
\]
7. If \( f \) is a continuous function and \( a_n \to L \), then

8. Squeeze Theorem:

Examples: Find the limits of the following sequences:

\[
a_n = \frac{\ln(n + e^{3n})}{n}
\]
\[ a_n = \left( 1 + \frac{3}{n} \right)^{n/2} \]

\[ a_n = \arctan \left( \frac{n}{n+1} \right) \]

\[ a_n = \frac{(-1)^{n+1}}{2n+1} \]
More Definitions:

Monotonic sequence:

\{a_n\} is increasing for \( n \geq N \) if and only if (Implications:)

1. \( a_{n+1} - a_n > 0 \);
2. \( \frac{a_{n+1}}{a_n} > 1 \) if \( a_n > 0 \);
3. If \( a_n = f(n) \) for some real-valued function \( f \), then \( f' > 0 \).

\{a_n\} is decreasing for \( n \geq N \) if and only if (Implications:)

\{a_n\} is bounded above (below) if and only if

Monotone Sequence Theorem:

Determine if the sequence \( a_n = \frac{\ln n}{n} \) is monotonic and bounded.

Given the sequence defined recursively by \( a_1 = 1, a_{n+1} = \sqrt{3 + a_n} \) is increasing and bounded above by 3, find the limit.
On Beyond Average:

Find the limit of \( a_n = (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n} + \frac{1}{2}}{n} \)

Given \( a_n = \frac{1000^n}{n!} \), show \( a_n \) is decreasing (for \( n \geq \text{some } N \)) and bounded below. What is the limit of this sequence, and why?
Proof by Induction (Appendix E)

Example:

Given \( a_0 = 1 \) and \( a_{n+1} = \frac{1}{3}a_n + 1 \):

a) Show \( a_n \) is increasing and \( 0 < a_n < 2 \) for all \( n \).

b) Find \( \lim_{n \to \infty} a_n \).