1 Overview of Convergence Tests

Examples: Determine whether the series converges absolutely, converges conditionally, or diverges. Name the test used and show all conditions are met.

\[
\sum_{n=1}^{\infty} \frac{n^{3}}{n^{2} + 5n^{4} + 7}
\]

Compare with \( \sum \frac{1}{n^{2}} \) convergent by \( p \)-Test

\[a_{n} = \frac{n^{3}}{n^{2} + 5n^{4} + 7} > 0\]

\[\left(\frac{n}{n^{2} + 5n^{4} + 7}\right)^{\frac{1}{2}} > 0\]

\[
\lim_{n \to \infty} \frac{a_{n}}{\left(\frac{n}{n^{2} + 5n^{4} + 7}\right)^{\frac{1}{2}}} = \lim_{n \to \infty} \frac{n^{3}}{\frac{n^{2} + 5n^{4} + 7}{n^{2} + 5n^{4} + 7}} = 1 > 0
\]

\[
\sum \frac{n^{3}}{n^{2} + 5n^{4} + 7}
\]

is convergent by LCT with \( \sum \frac{1}{n^{2}} \) (\( n^{2} \geq 0 \))
\[
\sum_{n=1}^{\infty} \frac{n^2}{n^2 + 1}
\]

Compare with \( \sum \frac{1}{n} \) divergent by p-test

\[
a_n = \frac{n}{n^2 + 1} > 0
\]

\[
b_n = \frac{1}{n} > 0
\]

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n}{n^2 + 1} \cdot \frac{n}{1} = \lim_{n \to \infty} \frac{n^2}{n^2 + 1} = 1 \geq 0
\]

\[
\therefore \sum \frac{n^2}{n^2} \text{ is divergent by LCT with } \sum \frac{1}{n}
\]
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + 3}
\]

\[b_n = 1/n = \frac{1}{n+3}\]  

**pos, dec**

\[f'(x) = -(x+3)^{-2} = \frac{-1}{(x+3)^2} < 0\]

\[\lim_{n \to \infty} \frac{1}{n+3} = 0\]

\[\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3} \text{ is convergent by RST}\]

**Abs conv?**

Test \[\sum_{n=1}^{\infty} \frac{1}{n+3}\]

Compare with \(\sum \frac{1}{n}\) divergent by P-Test

\[b_n = \frac{1}{n+3} > 0\]

\[c_n = \frac{1}{n} > 0\]

\[\lim_{n \to \infty} \frac{b_n}{c_n} = \lim_{n \to \infty} \frac{1}{n+3} \cdot \frac{n}{1} = \lim_{n \to \infty} \frac{n}{n+3} = 1\]

\[\therefore \sum \frac{1}{n+3} \text{ is divergent by LCT with } \sum \frac{1}{n}\]

\[\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+3} \text{ is conditionally convergent}\]
\[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n + 1)3^n}{2^{2n}}\]

**Ratio Test**

\[\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|\]

\[= \lim_{n \to \infty} \frac{(n+2)3^{n+1}}{2^{2(n+1)}} \cdot \frac{2^n}{(n+1)3^n}\]

\[= \lim_{n \to \infty} \frac{n+2}{n+1} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{2}{2^{2n+2}}\]

\[= 1 \cdot 3 \cdot \frac{1}{4}\]

\[= \frac{3}{4} < 1\]

\[\therefore \text{ series is absolutely convergent by Ratio Test}\]
\[ \sum_{n=0}^{\infty} \frac{2n^2}{1 + n^2} \]

\[ a_n = \frac{2n^2}{1 + n^2} \rightarrow 2 \neq 0 \]

\[ \therefore \text{series is divergent by Test for Divergence} \]
\[
\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^2}
\]

Compare with \(\sum \frac{1}{n^2}\) convergent by P-Test

\[
a_n = \frac{\sin\left(\frac{1}{n}\right)}{n^2} > 0
\]

\[
b_n = \frac{1}{n^2} > 0
\]

Let:

\[
\frac{a_n}{b_n} = \frac{\sin\left(\frac{1}{n}\right)}{n^2} \to 0
\]

We want:

\[
\frac{\sin\left(\frac{1}{n}\right)}{n^2} \leq \frac{1}{n^2}
\]

\[
\sin\left(\frac{1}{n}\right) \leq 1
\]

\[\therefore \text{series is (absolutely) convergent by Comparison with } \sum \frac{1}{n^2}\]
For the last series, $\lim_{n \to \infty} a_n = 1.037392940198405$. Estimate how close this is to the actual sum of the series.

\[
S - S_N \leq \sum_{n=N+1}^{\infty} f(n) \, dx \\
S - S_{100} \leq \int_{100}^{\infty} \frac{\sin\left(\frac{1}{x}\right)}{\pi x} \, dx \\
= \lim_{a \to \infty} \int_{100}^{a} \frac{\sin\left(\frac{1}{x}\right)}{\pi x} \, dx \\
= \lim_{a \to \infty} [\cos\left(\frac{1}{x}\right)]_{100}^{a} \\
= \lim_{a \to \infty} \left(\cos\left(\frac{1}{a}\right) - \cos\left(\frac{1}{100}\right)\right) \\
= 1 - \cos\left(\frac{1}{100}\right)
\]
How many terms are needed to find the sum of the third and fourth series correct to within 0.001?

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)3^n}{2^{2n}} \]

\[ |s - s_N| \leq b_{N+1} < 0.001 \]

\[ \frac{1}{N+4} < \frac{1}{1000} \]

\[ 1000 < N+4 \]

\[ N > 996 \]

(at least 997 terms)

(in math) At least 36 terms