Week 3 Review

Tuesday, February 4, 2020 5:00 PM

MATH 152-Spring 2020
Week in Review III

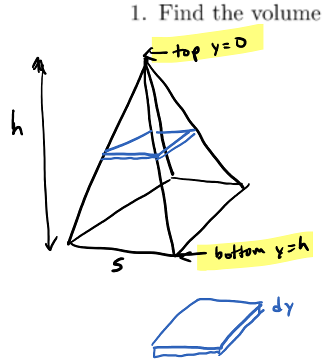
TEXAS A&M UNIVERSITY
Math Learning Center

(Problems with a * beside them will also be done in Python)

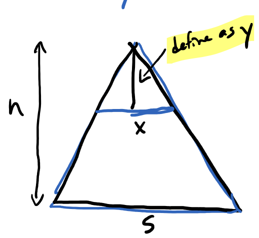
1 Section 6.2-6.3 Shells vs Slices

	f(x)	g(y)
about x-axis	Slices $V=\int(\pi R^2-\pi r^2)\,dx$	Shells $V=\int 2\pi r\,y(R-L)\,dy$
about y-axis	Shells $V=\int 2\pi x(T-B)\,dx$	Slices $V=\int(\pi R^2-\pi r^2)\,dy$

1. Find the volume of a square pyramid whose height is h and whose base is s by s. No rotation → slices



$V = x^2\ dy$



$\frac{y}{x} = \frac{h}{s}$

$x = \frac{s}{h}y$

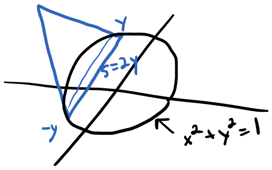
$$V = \int_0^h \left(\frac{s}{h}y\right)^2 dy$$

$$= \frac{s^2}{h^2}\int_0^h y^2\,dy$$

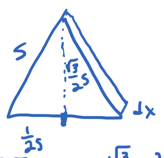
$$= \frac{s^2}{h^2}\left(\frac{1}{3}y^3\Big|_0^h\right)$$

$$= \frac{1}{3}\frac{s^2}{h^2}\cdot h^3 = \boxed{\frac{1}{3}s^2 h}$$

2. The base of a solid is the unit circle in the x-y plane. Cross-sections perpendicular to the x-axis are equilateral triangles. Find the volume of the solid.Find the volume of the solid formed by rotating the region bounded by $y = -2 + 3x - x^2$ and $y = 0$ about the y-axis.



$x^2+y^2=1$ so $y^2=1-x^2$



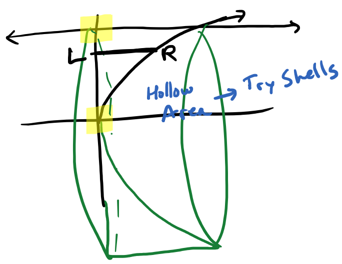
$$V = \int_{-1}^{1} \sqrt{3}\,(1-x^2)\,dx$$
$$= 2\int_0^1 \sqrt{3}\,(1-x^2)\,dx$$
$$= 2\sqrt{3}\left(x - \tfrac{1}{3}x^3\right)\Big|_0^1$$
$$= 2\sqrt{3}\left(1-\tfrac{1}{3}\right) = \boxed{\tfrac{4}{3}\sqrt{3}}$$

$$V = \tfrac{1}{2}\,s\left(\tfrac{\sqrt{3}}{2}s\right)dx = \tfrac{\sqrt{3}}{4}s^2\,dx = \tfrac{\sqrt{3}}{4}(2y)^2\,dx = \sqrt{3}\,y^2\,dx$$

(Now for rotations and our chart)

3. Find the volume of the solid formed by rotating the region bounded by the curves $y = \sqrt{x}$ ($x = y^2$), $x = 0$, and $y = 2$ about the x-axis.

Shells → about x axis → $g(y)$ (Opposite variable)

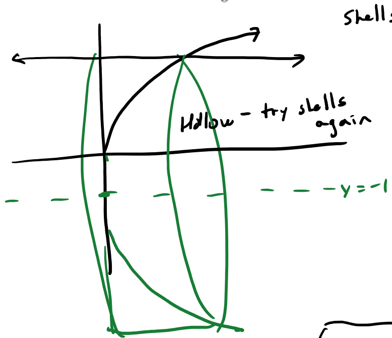


Always use the region to get boundaries

$$V = \int 2\pi y\,(\text{Right}-\text{Left})\,dy$$
$$= \int_0^2 2\pi y\,(y^2 - 0)\,dy$$
$$= 2\pi \int_0^2 y^3\,dy$$
$$= 2\pi\,\tfrac{1}{4}y^4\Big|_0^2$$
$$= \boxed{8\pi}$$

4. Find the volume of the solid formed by rotating the region in the previous example about the line $y = -1$. "x" axis

Shells ⇒ $g(y)$



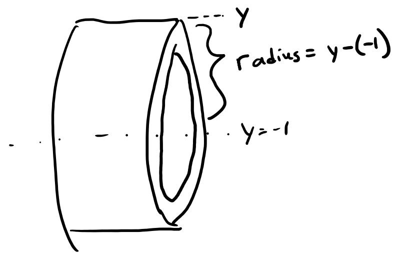
$$V = \int 2\pi(y+1)(R-L)\,dy$$

$$= \int_0^2 2\pi(y+1)(y^2-0)\,dy$$

$$= 2\pi \int_0^2 (y^3+y^2)\,dy$$

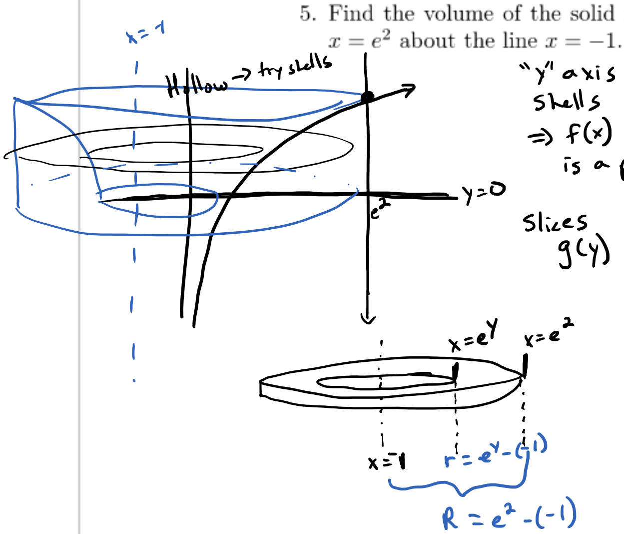
$$= 2\pi\left(\frac{1}{4}y^4 + \frac{1}{3}y^3\right|_0^2$$

$$= 2\pi\left(4+\frac{8}{3}\right) = \boxed{\frac{40\pi}{3}}$$



$x = e^y$

5. Find the volume of the solid formed by rotating the region bounded by $y = \ln x$, $y = 0$, and $x = e^2$ about the line $x = -1$. *



"y" axis

Shells

⇒ $f(x)$ but $y = \ln x$ is a problem ✗

Slices

$g(y)$

$$V = \int (\pi R^2 - \pi r^2)\,dy$$

$e^y = e^2$

$y = 2$

$$= \int_0^2 \left(\pi(e^2+1)^2 - \pi(e^y+1)^2\right)dy$$

(Set up only - integrate in Python)

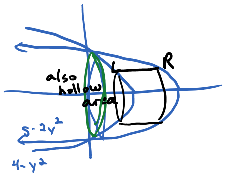
```
In [1]: from sympy import *

In [2]: y=symbols('y')
        Vol=integrate(pi*(E**2+1)**2-pi*(exp(y)+1)**2,(y,0,2))
        print('The volume is',Vol,'or approximately',Vol.evalf())

The volume is 5*pi/2 + 2*pi*exp(2) + 3*pi*exp(4)/2 or approximately 311.568510914280
```

6. Set up, but do not compute, an integral to find the volume of the solid obtained by rotating the region bounded by $x = 4 - y^2$ and $x = 8 - 2y^2$ about the x-axis.*

upper half only



g(y)
about x-axis
Shells

$$V = \int 2\pi\, y\,(R - L)\, dy$$

$$= \int_0^2 2\pi\, y\,\left((8-2y^2) - (4-y^2)\right)\, dy$$

(integrate in Python)

Boundaries:

$4-y^2 = 8-2y^2$

$y^2 = 4$

$y = \cancel{\pm}2$

In [3]:
```
y=symbols('y')
Vol=integrate(2*pi*y*((8-2*y**2)-(4-y**2)),(y,0,2))
print('The volume is',Vol,'or approximately',Vol.evalf())
```

```
The volume is 8*pi or approximately 25.1327412287183
```

2 Section 6.4

Work - a force exerted over a displacement

3 types of problems:

1) Hooke's Law $F = kx$, $W = \int F\, dx$

2) Pumping Liquid out of a tank $W = \int$ Volume × Density × Distance Traveled

(dy here)

3) Cable Lifting - can be done using either of the above strategies

Hooke's Law

1. A spring has a natural length of 3 meters. A force of 10 N is required to keep the spring stretched an additional 50 cm. Find the amount of work required to stretch the spring from its natural length to a length of 5m. ← (stretches 2m from natural length)

$F = kx$ use given information to find k, then integrate to find work

$10 = k \cdot (.5)$

$k = 20 \ \frac{N}{m}$

$W = \int_0^2 20x \, dx$

$= 10x^2 \Big|_0^2$

$= \boxed{40 \ N \cdot m \text{ or Joules}}$

2. A spring has a natural length of 3 meters. The work required to stretch the spring an additional 50 cm is 10 Joules. Find the amount of work required to stretch the spring from its natural length to a length of 5m.

$F = kx$

$W = \int_0^{\frac{1}{2}} kx \, dx = 10$

$\frac{1}{2}kx^2 \Big|_0^{1/2} = 10$

$\frac{1}{8}k = 10$

$k = 80 \ \frac{N}{m}$

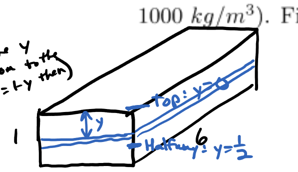
$W = \int_0^2 80x \, dx$

$= 40x^2 \Big|_0^2$

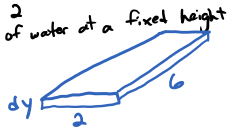
$= \boxed{160 \ J}$

3. A tank in the shape of a rectangular prism 6m long, 2m wide, and 1m tall is full of water (density 1000 kg/m^3). Find the work required to pump half of the water out of the tank.

Can also define y from the bottom to the slice. (dist = 1-y then)



Idea: a thin slice of water at a fixed height



$F = \text{volume} \times \text{density}$

$= 12\,dy \cdot \rho g$

$W = F \times \text{dist}$

$= 12 \rho g\, dy \cdot y$

$$W = \int_0^{\frac{1}{2}} 12\rho g\, y\, dy$$

$$= 12\rho g \left(\tfrac{1}{2} y^2 \right)\Big|_0^{1/2}$$

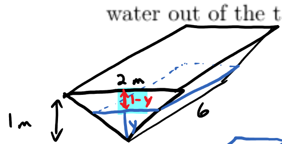
$$= \tfrac{3}{2} \rho g$$

Online HW: in metric, $\rho = 1000$ $g = 9.8$

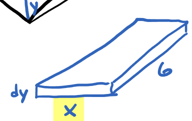
in English, $\rho g = 62.5$

$$= \tfrac{3}{2}(9800) = \boxed{14{,}700 \text{ J}}$$

4. Suppose the tank in the previous problem instead has ends in the shape of isosceles triangles with a height of 1m and a base across the top of 2m. Find the work required to pump all the water out of the tank.



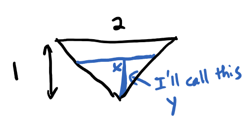
width changes top to bottom



$F = \text{Vol} \times \text{Dens}$

$= 6x\, dy \cdot \rho g$

Again, need x in terms of y



$\frac{y}{x} = \frac{1}{2} \rightarrow x = 2y$

$W = F \cdot \text{dist}$

$= 6(2y)\rho g\, dy \cdot (1-y)$

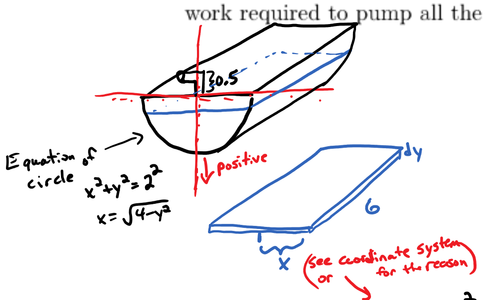
$$W = \int_0^1 12\rho g\, y(1-y)\, dy$$

$$= 12\rho g \int_0^1 (y - y^2)\, dy$$

$$= 12\rho g \left(\tfrac{1}{2} y^2 - \tfrac{1}{3} y^3 \right)\Big|_0^1$$

$$= 2\rho g = \boxed{19{,}600 \text{ J}}$$

5. Now suppose the tank in #3 has ends in the shape of a semicircle (lower half) of radius 2m. A spout extends 0.5m above the top of the tank. If the tank is full of water, find the amount of work required to pump all the water out of the tank.*



$V = 6(2x)\,dy$

$F = 12\sqrt{4-y^2}\cdot dy \cdot \rho g$

$W = 12\rho g\sqrt{4-y^2}\,dy \cdot (y+\frac{1}{2})$ ← spout

$$W = \int_0^2 12\rho g\sqrt{4-y^2}\,(y+\tfrac{1}{2})\,dy$$

By hand:

$$= 12\rho g\left(\int_0^2 y\sqrt{4-y^2} + \int_0^2 \tfrac{1}{2}\sqrt{4-y^2}\,dy\right)$$

u subs $u=4-y^2$

$\int_0^2\sqrt{4-y^2}\,dy$ is Area quarter-circle!

$A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(2)^2 = \pi$

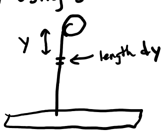
In Python:

```
In [5]: y=symbols('y')
        W=12*(9800)*integrate(sqrt(4-y**2)*(y+Rational(1,2)),(y,0,2))
        print('The Work required is',W,'or approximately',W.evalf())

The Work required is 58800*pi + 313600 or approximately 498325.648031080
```

6. An 800-lb steel beam hangs from a 50-foot cable which weighs 6 pounds/foot. Find the work done in winding 20 feet of the cable about a steel drum.

(I) Using slices



$F = \text{length} \times \text{density}$

$= dy \times 6$

$W = F \times \text{dist}$

$= 6\,dy \times y$

$W_1 = \int_0^{20} 6y\,dy$

$= 3y^2\Big|_0^{20}$

$= 1200$ ft-lbs

BUT we have a beam and another 30ft of cable that all moved 20 feet!

$W_2 = [800 + 30(6)](20)$

$= 19600$ ft lbs

Total

$W = 1200 + 19600$

$= \boxed{20{,}800 \text{ ft-lbs}}$

(II) Write the weight as a function of y, the amount of cable already pulled up.

Total weight $= 800 + (50)(6) = 1100$ lbs

Each ft of cable reduces the weight by 6 lbs,

So $F(y) = 1100 - 6y$

$$W = \int F\,dy = \int_0^{20}(1100-6y)\,dy$$

$= 1100y - 3y^2\Big|_0^{20}$

$= 22000 - 1200$

$= \boxed{20{,}800 \text{ ft-lbs}}$