Week 4 Review

Tuesday, February 11, 2020 4:06 PM

1 Section 7.1 Integration by Parts

Based on Product Rule: $d(uv) = u\,dv + v\,du$

$$\int u\,dv = uv - \int v\,du$$

Idea: Choose a part u to differentiate
part dv to integrate

(when possible, x^n should be u)

1. Evaluate the following integrals:

Product - Substitution will not work → IBP

(a) $\int x \cdot \cos x \, dx$

Let $u = x$ $\quad dv = \cos x \, dx$

Then $du = 1\,dx$ $\quad v = \sin x$

$$\int u \; dv = u \; v - \int v \; du$$

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

Now able to directly integrate

$$= x \sin x + \cos x + C$$

Cannot substitute since $2x$ is in exponent! Try IBP

(b) $\int_0^1 x^2 e^{-2x}\,dx$

Let $u = x^2$ $\quad dv = e^{-2x}\,dx$

$du = 2x\,dx$ $\quad v = -\frac{1}{2}e^{-2x}$

$$\int x^2 e^{-2x}\,dx = x^2\left(-\tfrac{1}{2}e^{-2x}\right) + \int \tfrac{1}{\cancel{2}} e^{-2x} \cdot \cancel{2}x\,dx$$

$\int u\,dv = u\,v - \int v\,du$

Cannot directly integrate, but improved!

$$= -\tfrac{1}{2}x^2 e^{-2x} + \int x e^{-2x}\,dx$$

ALWAYS Keep parts the same when using IBP twice or more!

Let $u = x$ $\quad dv = e^{-2x}\,dx$

Then $du = 1\,dx$ $\quad v = -\frac{1}{2}e^{-2x}$

$$= -\tfrac{1}{2}x^2 e^{-2x} + x\left(-\tfrac{1}{2}e^{-2x}\right) + \int \tfrac{1}{2}e^{-2x}\,dx$$

Can directly integrate now

$$\int_0^1 x^2 e^{-2x}\,dx = -\tfrac{1}{2}x^2 e^{-2x} - \tfrac{1}{2}x e^{-2x} - \tfrac{1}{4}e^{-2x}\Big|_0^1$$

$$= \left(-\tfrac{1}{2}e^{-2} - \tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-2}\right) - \left(0 + 0 - \tfrac{1}{4}\right)$$

$$= \boxed{\tfrac{1}{4} - \tfrac{5}{4}e^{-2}}$$

Product - cannot substitute

(c) $\int x^2 \ln x\,dx$

Problem with $u = x^2$ $\quad dv = \ln x\,dx$ CANNOT integrate $\ln x$ unless we do it by parts as well!

So we must let $u = \ln x$ $\quad dv = x^2\,dx$

then $du = \frac{1}{x}\,dx$ $\quad v = \frac{1}{3}x^3$

$$\int x^2 \ln x\,dx = (\ln x)\left(\tfrac{1}{3}x^3\right) - \int\left(\tfrac{1}{3}x^{\cancel{3}\,2}\right)\left(\tfrac{1}{\cancel{x}}\right)dx$$

Cancel x's, then integrate directly

$$= \tfrac{1}{3}x^3 \ln x - \tfrac{1}{3}\int x^2\,dx$$

$$= \boxed{\tfrac{1}{3}x^3 \ln x - \tfrac{1}{9}x^3 + C}$$

2 Section 7.2 Trig Integrals

Basic Functions

$\int \sin x \, dx = -\cos x + C$ $\int \cos x \, dx = \sin x + C$

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| = \ln|\sec x| + C$

$u = \cos x$
$du = -\sin x \, dx$

Similar technique $\int \cot x \, dx = \ln|\sin x| + C$

$\int \sec x \, dx = \ln|\sec x + \tan x| + C$ $\int \csc x \, dx = \ln|\csc x - \cot x| + C$

Squares

$\int \sin^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) dx$

$\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2x) dx$

Based on: $c^2 + s^2 = 1$
$c^2 - s^2 = \cos(2x)$
+ or −

Most General Powers: Strategy: set aside a "du" and use identities to turn the rest of the integral into "u".

1. Evaluate the following integrals: Set aside $du = \cos x$ ($u = \sin x$)
or $du = -\sin x$ ($u = \cos x$) ~~X~~ want to leave even powers to change

(a) $\displaystyle\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx$

turn into sines* Set aside

* $\cos^2 x = 1 - \sin^2 x$ (This is why you want even powers)

$= \int_0^{\pi/2} \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx$

if $x = \frac{\pi}{2}$ $u = \sin\left(\frac{\pi}{2}\right) = 1$
if $x = 0$ $u = \sin 0 = 0$

$= \int_0^{\pi/2} \sin^2 x (1 - \sin^2 x) \cdot \cos x \, dx$

Let $u = \sin x$
$du = \cos x \, dx$
$dx = \frac{du}{\cos x}$

$= \int_0^1 u^2(1 - u^2) \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$

$= \int_0^1 (u^2 - u^4) \, du$

$= \frac{1}{3}u^3 - \frac{1}{5}u^5 \Big|_0^1$

$= \frac{1}{3} - \frac{1}{5} = \boxed{\frac{2}{15}}$

Set aside $du = \sec^2 x$ $(u = \tan x)$

OR $du = \sec x \tan x$ $(u = \sec x)$

Want even powers left

(b) $\displaystyle\int_0^{\pi/4} \tan^2 x \sec^4 x \, dx$

turn into tangent

Set aside

$$= \int_0^{\pi/4} \tan^2 x \ \sec^2 x \ \sec^2 x \, dx$$

$$\frac{s^2}{c^2} + \frac{c^2}{c^2} = \frac{1}{c^2}$$

$$\tan^2 + 1 = \sec^2$$

$$= \int_0^{\pi/4} \tan^2 x \left(\tan^2 x + 1\right) \sec^2 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$dx = \dfrac{du}{\sec^2 x}$

if $x = \pi/4$, $u = \tan\left(\frac{\pi}{4}\right) = 1$

if $x = 0$, $u = \tan(0) = 0$

$$= \int_0^1 u^2 \left(u^2 + 1\right) \cancel{\sec^2 x} \frac{du}{\cancel{\sec^2 x}}$$

$$= \int_0^1 \left(u^4 + u^2\right) du$$

$$= \frac{1}{5}u^5 + \frac{1}{3}u^3 \Big|_0^1$$

$$= \left(\frac{1}{5} + \frac{1}{3}\right) - 0 = \boxed{\frac{8}{15}}$$

CANNOT save a cos x since leaves odd powers to change!

(c) $\displaystyle\int \cos^4 x \, dx$

use identity from 7.2 screen

$$= \int \cos^2 x \cdot \cos^2 x \, dx$$

$$= \int \frac{1}{2}\left(1 + \cos 2x\right) \cdot \frac{1}{2}\left(1 + \cos 2x\right) dx$$

$$= \frac{1}{4} \int \left(1 + 2\cos 2x + \cos^2 2x\right) dx$$

$u = 2x$

$du = 2\,dx$

$$= \frac{1}{4}\left(x + \sin 2x + \int \cos^2 2x \, dx\right)$$

use identity again

$$= \frac{1}{4}\left(x + \sin 2x + \int \frac{1}{2}\left(1 + \cos 4x\right) dx\right)$$

$$= \boxed{\frac{1}{4}\left(x + \sin 2x + \frac{1}{2}x + \frac{1}{8}\sin 4x\right) + C}$$

3 Exam I Review

1. Evaluate the following integrals: $\frac{1}{x}\cdot(\ln x)^{\frac{1}{2}}$ Product, BUT $d(\ln x)=\frac{1}{x}$ so use substitution

(a) $\displaystyle\int_1^e \frac{\sqrt{\ln(x)}}{x}\,dx$

if $x=e$ $u=\ln(e)=1$
if $x=1$ $u=\ln(1)=0$

$u=\ln x$
$du=\frac{1}{x}\,dx$
$dx=x\cdot du$

$= \int_0^1 \frac{\sqrt{u}}{\cancel{x}}\cdot\cancel{x}\,du$

$= \frac{2}{3}u^{3/2}\Big|_0^1$

$= \frac{2}{3}-0=\boxed{\frac{2}{3}}$

$x\cdot(1-9x^2)^{1/2}$ $d(1-9x^2)=-18x$ so use substitution

(b) $\displaystyle\int x\sqrt{1-9x^2}\,dx$

Let $u=1-9x^2$
$du=-18x\,dx$
$dx=\frac{du}{-18x}$

$=\int \cancel{x}\,u^{1/2}\cdot\frac{du}{-18\cancel{x}}$

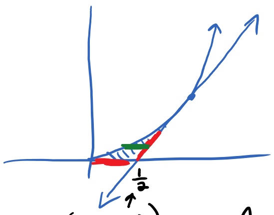
$=-\frac{1}{18}\int u^{1/2}\,du$

$=\frac{-1}{27}u^{3/2}+C$

$=-\frac{1}{27}(1-9x^2)^{3/2}+C$

2. Find the area of the regions bounded by the following curves:

(a) The parabola $y = x^2$, the x-axis, and the line tangent to the parabola at $x = 1$.



Tangent Line: $L(x) = f(a) + f'(a)(x-a)$
$= 1^2 + 2(1)(x-1)$
$y = 2x - 1$

Option 1: T-B (need to split into 2 integrals)

$$A = \int_0^{\frac{1}{2}} (x^2 - 0)\,dx + \int_{\frac{1}{2}}^1 (x^2 - (2x-1))\,dx$$

⋮

Option 2 R-L (need to change to g(y))

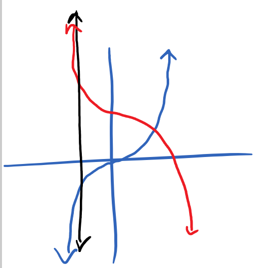
$y = x^2 \rightarrow x = \sqrt{y}$

$y = 2x - 1 \rightarrow x = \frac{1}{2}(y+1)$

$$A = \int_0^1 \left(\frac{1}{2}(y+1) - y^{1/2}\right) dy$$

⋮

(b) $y = x^3$ and $y = 16 - x^3$ and $x = -2$ (Not enclosed otherwise)



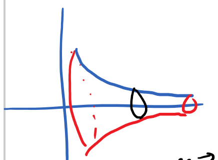
Intersection: $x^3 = 16 - x^3$
$2x^3 = 16$
$x^3 = 8$
$x = 2$

T - B

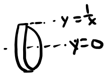
$$A = \int_{-2}^{2} ((16 - x^3) - x^3)\,dx$$
$$= \int_{-2}^{2} (16 - 2x^3)\,dx$$
$$= 16x - \frac{1}{4}x^4 \Big|_{-2}^{2}$$
$$= (32 - 4) - (-32 - 4) = \boxed{64}$$

3. Find the volumes of the solids described below:

(a) Formed by rotating the region bounded by $y = \frac{1}{x}$, $y = 0$, $x = 1$, and $x = 5$ about the x-axis.



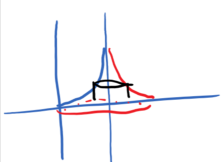
No hollow area → Try slicing first



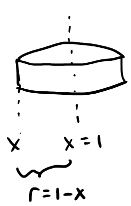
	$f(x)$	$g(y)$
about x-axis	Slice $\int(\pi R^2 - \pi r^2)\,dx$	Shell $\int 2\pi y(R-L)\,dy$
about y-axis	Shell $\int 2\pi x(T-B)\,dx$	Slice $\int(\pi R^2 - \pi r^2)\,dy$

$$V = \int_1^5 \pi\left(\left(\tfrac{1}{x}\right)^2 - 0^2\right)dx$$
$$= \pi \int_1^5 x^{-2}\,dx$$
$$= -\pi\, x^{-1} \Big|_1^5$$
$$= -\pi\left(\tfrac{1}{5} - 1\right) = \boxed{\tfrac{4\pi}{5}}$$

(b) Formed by rotating the region bounded by the x-axis, $x = 1$, and $y = x^3$ about the line $x = 1$.



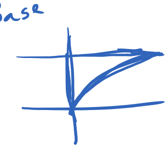
No hollow area, but $f(x)$ looks easier (x^3 vs $y^{1/3}$)
about "y" axis ⇒ Shells



$$V = \int_0^1 2\pi\,(1-x)\,\underbrace{(x^3 - 0)}_{T\,-\,B}\,dx$$
$$= 2\pi\int_0^1 (x^3 - x^4)\,dx$$
$$= 2\pi\left(\tfrac{1}{4}x^4 - \tfrac{1}{5}x^5\right)\Big|_0^1$$
$$= 2\pi\left(\tfrac{1}{4} - \tfrac{1}{5}\right) = \boxed{\tfrac{\pi}{10}}$$

(c) Base of the solid is the region enclosed by the y-axis, $y = 1$, and $y = \sqrt{x}$. Cross-sections perpendicular to the y-axis are semicircles.

Base



$y = \sqrt{x}$

So $x = y^2$

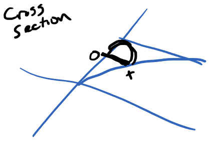
$$V = \int_0^1 \frac{1}{2}\pi\left(\frac{1}{2}\cdot y^2\right)^2 dy$$

$$= \frac{\pi}{8}\int_0^1 y^4\, dy$$

$$= \frac{\pi}{8}\left(\frac{1}{5}y^5\right|_0^1$$

$$= \boxed{\frac{\pi}{40}}$$

Cross section



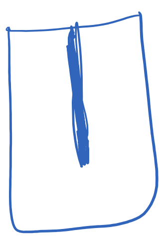
☆ So $r = \frac{1}{2}x$ ☆

$V = \frac{1}{2}\pi r^2\, dy$

$= \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 dy$

density $= \frac{25}{50} = \frac{1}{2}$ lb/ft

4. A 50-foot rope that weighs 25 pounds hangs from the top of a large building. How much work is required to pull 10 feet of rope to the top?



Easiest: Write weight as a function of y, the amount of rope pulled to the top

$F = 25 - \frac{1}{2}y$ every ft of rope reduces weight by $\frac{1}{2}$ lb.

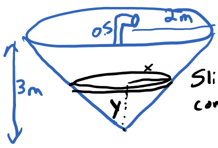
$$\text{Work} = \int_a^b F(y)\, dy$$

$$= \int_0^{10}\left(25 - \frac{1}{2}y\right) dy$$

$$= 25y - \frac{1}{4}y^2\Big|_0^{10}$$

$$= 250 - 25 = \boxed{225 \text{ ft-lbs}}$$

5. A conical tank is 3m tall, has a 2m radius across the top, and has a 0.5m spout extending from the top. If the tank is full of water, find the work required to pump all the water out of the tank (use ρg for the weight density of the water).



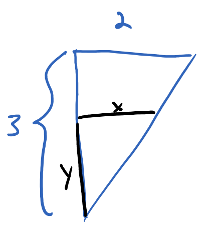
Slice water at constant height

$$W = \int_0^3 \underbrace{\rho g \pi \left(\tfrac{2}{3}y\right)^2 dy}_{F} \cdot \underbrace{(3.5-y)}_{\text{dist}}$$

$$= \rho g \pi \int_0^3 \frac{4}{9} y^2 \left(\frac{7}{2}-y\right) dy$$

$\vdots$

$$F = \text{Vol} \times \text{Density}$$
$$= \pi x^2 \, dy \cdot \rho g$$



$$\frac{x}{y} = \frac{2}{3}$$

$$x = \frac{2}{3}y$$

(NOTE: can also define y from top to slice)