12.3-Partial Derivatives

Given a function of more than one variable, the partial derivative of the function with respect to one variable is found by treating all the other variables as constants. This is evidenced by the (two-variable) definitions:

\[
\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \quad y \text{ is constant}
\]

\[
f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h} \quad x \text{ is constant}
\]

What partial derivatives give us:

1) The rate of change in \( f \) as one variable changes (others constant)

2) Graphically: \( \langle 1, 0, f_x \rangle \) are vectors tangent to surface \( z = f(x, y) \)

\( \langle 0, 1, f_y \rangle \)
Partial Differential Equations: An equation relating an unknown multivariate function and its partial derivatives

Examples:

Given \( f(x,y) = x\sqrt{y} - \frac{y}{\sqrt{x}} \), find \( \frac{\partial f}{\partial x} \) and \( f_y(1,1) \)

\[
\frac{\partial f}{\partial x} = \sqrt{y} + \frac{1}{2} x^{-\frac{3}{2}} y
\]

\[
f_y(1,1) = \frac{1}{2} x \cdot y^{-\frac{1}{2}} - x^{-\frac{3}{2}}
\]

\[
f_y(1,1) = \frac{1}{2} \cdot 1 \cdot 1^{-\frac{1}{2}} - 1^{-\frac{3}{2}} = \frac{-1}{2}
\]

**Alt** \( x = 1 \)

\[
f(1,y) = \sqrt{y} - y
\]

\[
f_y(1,1) = \frac{1}{2} y^{-\frac{1}{2}} - 1
\]

\[
= \frac{1}{2} - 1 = \frac{-1}{2}
\]
Given \( f(x, y) = e^{xy^2} \), find all the second partial derivatives.

\[
\begin{align*}
    f_x(x, y) &= y^2 e^{xy^2} \\
    f_y(x, y) &= 2xy e^{xy^2} \\
    f_{xx} &= \frac{\partial}{\partial x}(y^2 e^{xy^2}) = y^4 e^{xy^2} \\
    f_{xy} &= \frac{\partial}{\partial y}(y^2 e^{xy^2}) = y^2(2xy e^{xy^2}) + e^{xy^2}(2y) = 2y e^{xy^2}(xy^2 + 1) \\
    f_{yx} &= \frac{\partial}{\partial x}(2xy e^{xy^2}) = 2xy(y^2 e^{xy^2}) + e^{xy^2}(2y) = 2y e^{xy^2}(xy^2 + 1) \\
    f_{yy} &= \frac{\partial}{\partial y}(2xy e^{xy^2}) = 2xy(2xy e^{xy^2}) + e^{xy^2}(2x) = 2x e^{xy^2}(2xy^2 + 1)
\end{align*}
\]
Show that the function $u = \frac{t}{4t^2 - x^2}$ is a solution to the wave equation $u_{tt} = 4u_{xx}$.

$u_t = \frac{(4t^2 - x^2)(1) - t(8t)}{(4t^2 - x^2)^2} = \frac{-4t^2 - x^2}{(4t^2 - x^2)^2}$

$u_{xx} = \frac{(4t^2 - x^2)(2t) - 2xt(2)(4t^2 - x^2)(-2x)}{(4t^2 - x^2)^3} = \frac{8t^3 + 6tx^2}{(4t^2 - x^2)^3}$

$4u_{xx} = u_{tt} \checkmark$
Review of Derivative Rules:

\[
\begin{align*}
\frac{d}{dx}(x^n) &= nx^{n-1} \\
\frac{d}{dx}(uv) &= u \frac{dv}{dx} + v \frac{du}{dx} \\
\frac{d}{dx}\left(\frac{u}{v}\right) &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \\
\frac{d}{dx}(\sin x) &= \cos x \\
\frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\tan x) &= \sec^2 x \\
\frac{d}{dx}(\cot x) &= -\csc^2 x \\
\frac{d}{dx}(\sec x) &= \sec x \tan x \\
\frac{d}{dx}(\csc x) &= -\csc x \cot x \\
\frac{d}{dx}(f(g)) &= f'(g) \cdot g' \\
\frac{d}{dx}(e^x) &= e^x \\
\frac{d}{dx}(a^x) &= a^x \ln a \\
\frac{d}{dx}(\ln x) &= \frac{1}{x} \\
\frac{d}{dx}(\log_a x) &= \frac{1}{x \ln a} \\
\frac{d}{dx}(\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}(\cos^{-1} x) &= -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}(\tan^{-1} x) &= \frac{1}{1+x^2} \\
\frac{d}{dx}(\cot^{-1} x) &= -\frac{1}{1+x^2} \\
\frac{d}{dx}(\sec^{-1} x) &= \frac{1}{x \sqrt{x^2-1}} \\
\frac{d}{dx}(\csc^{-1} x) &= -\frac{1}{x \sqrt{x^2-1}}
\end{align*}
\]