Panel 1

12.5 The Chain Rule

One-variable Chain Rule: \( y = f(x) \); \( u = g(x) \), then \( \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \)

Two-variable Differentials: \( dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy \)

Generic Chain Rule: if \( z = f(x_1, x_2, x_3, \ldots, x_n) \) and \( x_i = g_i(t_1, t_2, \ldots, t_m) \), then

\[
\frac{\partial z}{\partial t_j} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_j} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_j} + \frac{\partial z}{\partial x_3} \cdot \frac{\partial x_3}{\partial t_j} + \ldots + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_j}
\]

(replace \( i \) with \( d \) where appropriate)

Given \( w \) is a function of \( x, y, z, \) and \( r \), and each of these variables are functions of \( s \) and \( t \), write the Chain Rule for finding \( \frac{\partial w}{\partial s} \):

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s} + \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial s}
\]

Panel 2

Implicit Differentiation

Recall: Given \( f(x, y) = 0 \), find \( \frac{dy}{dx} \)

If \( z = f(x, y) \), then \( \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} \)

Derivative of \( f(x, y) = f_x + f_y \cdot \frac{dy}{dx} = 0 \)

Solve for \( \frac{dy}{dx} \):

\[
\frac{dy}{dx} = -\frac{f_x}{f_y}
\]

\( f(x, y, z) = 0 \)

Find \( \frac{dz}{dx} : f_x + f_z \cdot \frac{dz}{dx} = 0 \)

\( \frac{dz}{dx} = -\frac{f_x}{f_z} \) etc.
Panel 3

Examples:

If \( z = x^2 y^3, x = 1 + \sqrt{t}, y = 1 - \sqrt{t} \), find \( \frac{dz}{dt} \)

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}
\]

\[
= 2xy^3 \left( \frac{1}{2} t^{-\frac{1}{2}} \right) + 3x^2 y^2 \left( -\frac{1}{2} t^{-\frac{1}{2}} \right)
\]

\[
= 2(1+\sqrt{t})(1-\sqrt{t})^3 \left( \frac{1}{2} t^{-\frac{1}{2}} \right) + 3(1+\sqrt{t})^2 (1-\sqrt{t})^2 \left( -\frac{1}{2} t^{-\frac{1}{2}} \right)
\]

If \( u = xy + yz + xz, x = st, y = e^{st}, \) and \( z = t^2 \), find \( u_x(0,1) \) and \( u_t(0,1) \)

Panel 4

\( u = xy + yz + xz \quad x = st \quad y = e^{st} \quad z = t^2 \)

\[
u_s = \frac{\partial u}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial u}{\partial y} \cdot \frac{dy}{ds} + \frac{\partial u}{\partial z} \cdot \frac{dz}{ds}
\]

\[
\frac{\partial u}{\partial s} = (y+z)(t) + (x+z) e^{st} \cdot t + (y+x)(0)
\]

\( u_s(0,1) = 2(1) + (1)(1) + 1(0) = 3 \)

\[
u_t = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}
\]

\[
= (y+z) \cdot 0 + (x+z) e^{st} \cdot 0 + (y+x) \cdot 2t
\]

\( u_t(0,1) = 2 \cdot 0 + 1(1)(0) + 1(2) = 2 \)
Panel 5

Find \( \frac{dy}{dx} \) if \( y^5 + 3x^2y^2 + 5x^4 = 12 \)

\[ f(x, y) = y^5 + 3x^2y^2 + 5x^4 - 12 = 0 \]

\[ \frac{df}{dx} \cdot \frac{dx}{dx} + \frac{df}{dy} \cdot \frac{dy}{dx} = 0 \]

\[ (4x^2 + 20x^3)(1) + (5y^4 + 6x^2y) \frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} = -\frac{4x^2 + 20x^3}{5y^4 + 6x^2y} \]

A block of ice is melting in the sun. Because of the way the sun is shining on it, the height is decreasing at a rate of 2 cm/min and the length and width are decreasing at 1 cm/min. How fast is the volume decreasing when the block has a length of 2 m, width of 3 m, and height of 3 m?

Panel 6

\[ V = lh^2 \]

\[ \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} + \frac{dV}{dl} \cdot \frac{dl}{dt} + \frac{dV}{lw} \cdot \frac{dw}{dt} \]

\[ \frac{dV}{dt} = -2L \frac{dh}{dt} + 2l \frac{dl}{dt} + Lw \frac{dw}{dt} \]

\[ L = 2 \text{ m}, \quad W = 3 \text{ m}, \quad H = 3 \text{ m} \]

\[ \frac{dL}{dt} = -0.1 \]

\[ \frac{dW}{dt} = -0.1 \]

\[ \frac{dH}{dt} = -0.2 \]

\[ \frac{dV}{dt} = -2 \cdot 2 \cdot (-0.1) + 2 \cdot 3 \cdot (-0.1) + 3 \cdot 3 \cdot (-0.2) \]

\[ \frac{dV}{dt} = -0.27 \text{ m}^3 \text{ min}^{-1} \]

On Your Own: 12.5 #1, 3, 5, 9, 13, 17, 21, 23, 31, 33, 39, 43