13.6 Applications of Double Integrals
(Moments + Center of Mass)

Given a 2-dimensional region (lamina) with density \( \rho(x,y) \)
find moments (about axes) and C.O.M.

**General Formula:**

\[
\text{moment} = \text{mass} \times \text{distance}
\]

(about axis) (to axis)

\[
\begin{align*}
M_x &= \rho (x_i, y_i) \Delta x_i \Delta y_j \cdot y_j^* \\
M_y &= \rho (x_i, y_i) \Delta x_i \Delta y_j \cdot x_i^*
\end{align*}
\]

\[
\begin{align*}
M_x &= \iint_D y \rho(x,y) \, dA \\
M_y &= \iint_D x \rho(x,y) \, dA \\
m &= \iint_D \rho(x,y) \, dA \\
(\bar{x}, \bar{y}) &= \left( \frac{M_y}{m}, \frac{M_x}{m} \right)
\end{align*}
\]
A circular region of radius 1 is placed 1 unit above an electric source. The charge density at any point is inversely proportional to the distance from the origin. Find the "center of charge" of the region.

\[
\rho(x,y) = \frac{1}{\sqrt{x^2+y^2}}
\]

Polar:

\[
x^2 + (y-1)^2 = 1
\]
\[
x^2 + y^2 - 2y = 0
\]
\[
r = 2\sin\theta
\]

Let \(u = \cos\theta\)
\[
du = -\sin\theta\,d\theta
\]

\[
M_x = \iint_0^\pi \int_0^{2\sin\theta} r\sin\theta \, r\,dr\,d\theta = \int_0^\pi \sin\theta \, d\theta = 0
\]
\[
M_y = \iint_0^\pi \int_0^{2\sin\theta} r\cos\theta \, r\,dr\,d\theta = \int_0^\pi \cos\theta \, d\theta = 0
\]
\[
m = \iint_0^\pi \int_0^{2\sin\theta} \, r\,dr\,d\theta = \int_0^\pi 2\sin\theta \, d\theta = \pi
\]

So \((\bar{x}, \bar{y}) = (0, \frac{8\pi}{4}) = (0, \frac{\pi}{2})\)