**Green's Theorem**

Let \( C \) be a positively-oriented, simple, piecewise smooth curve that encloses a region \( D \) in the \( x-y \) plane. If \( P \) and \( Q \) have continuous partial derivatives, then 

\[
\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]


\[
\text{oriented} = \text{counter clockwise}
\]

\[
P: \text{ piecewise smooth = differentiable except at corners}
\]

**Proof** (simple case where \( D \) consists of type 1 regions)

Claim: 

\[
\oint_C P \, dx = -\iint_D \frac{\partial P}{\partial y} \, dA
\]

\[
-\iint_D \frac{\partial P}{\partial y} \, dA = \int_a^b \int_{\alpha(x)}^{\beta(x)} \frac{\partial P}{\partial y} \, dy \, dx
\]

\[
= \int_a^b \left[ P(x, \beta(x)) - P(x, \alpha(x)) \right] \, dx
\]

(since \( x \) is constant)

Now 

\[
\oint_C P \, dx = \oint_{C_1} P \, dx + \oint_{C_2} P \, dx + \oint_{C_3} P \, dx + \oint_{C_4} P \, dx
\]

\( C_1: x=t, y=\beta(t) \quad C_2: x=t, y=T(t) \)

\[
= \int_a^b P(t, \beta(t)) \, dt - \int_a^b P(t, T(t)) \, dt
\]

\[
= -\int_a^b \left[ P(t, T(t)) - P(t, \beta(t)) \right] \, dt = -\iint_D \frac{\partial P}{\partial y} \, dA
\]

Similarly, 

\[
\oint_C Q \, dy = \iint_D \frac{\partial Q}{\partial x} \, dA
\]

\[
\therefore \oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]
Areas Using Line Integrals:
Recall \( \text{Area} = \iint_R 1 \, dA \)

We need \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \)

3 ways
① Let \( Q = x \) and \( P = 0 \)

② Let \( P = -y \) and \( Q = 0 \)

③ Let \( Q = \frac{1}{2}x \) and \( P = -\frac{1}{2}y \)

Green's Thm:
\[
\oint P \, dx + Q \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

Area:
\[
\begin{align*}
\text{Area} &= \int_C x \, dy \\
\text{Area} &= \int_C -y \, dx \\
\text{Area} &= \int_C \frac{1}{2}y \, dx + \frac{1}{2}x \, dy
\end{align*}
\]
Example: Compute \( \int_C x^2 y \, dx + Q \, dy \) where \( C \) is the triangular path from \((0,0)\) to \((1,0)\) to \((1,2)\) back to \((0,0)\). It is a closed curve, counterclockwise, so apply Green's Theorem:

\[
\int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

\[
= \int_0^1 \int_0^{2x} (1 - x^2) \, dy \, dx
\]

\[
= \int_0^1 (1 - x^2)(2x - 0) \, dx
\]

\[
= \int_0^1 (2x - 2x^3) \, dx
\]

\[
= 1 - \frac{1}{2} = \frac{1}{2}
\]
Ex. Suppose a particle travels one revolution counter-clockwise around the unit circle under the force field \( \vec{F}(x,y) = (e^x - y^3) \hat{i} + (\cos y + x^3) \hat{j} \). Find the work done by the field. 

Closed curve, counterclockwise \( \Rightarrow \) Green's Thm 

\[
W = \oint_C \vec{F} \cdot d\vec{r} = \int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

\[
= \iint_D \left( 3x^2 + 3y^2 \right) \, dA \quad \text{Polar Coordinates}
\]

\[
= \iint_0^1 \int_0^{2\pi} 3r^2 \cdot r \, dr \, d\theta
\]

\[
= 2\pi \int_0^1 3r^3 \, dr
\]

\[
= 2\pi \left( \frac{3}{4} \right) = \frac{3\pi}{2}
\]
Ex. Find the area enclosed by the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

Recall \( A = \int_C x \, dy = \int_C -y \, dx = \int_C \frac{1}{2} y \, dx + \frac{1}{2} x \, dy \)

Parametrize ellipse: \( x = a \cos t \)
\( y = b \sin t \), \( 0 \leq t \leq 2\pi \)

\[ A = \int_C x \, dy = \int_0^{2\pi} a \cos (b \cos t) \, dt \]
\[ = ab \int_0^{2\pi} \cos^2 t \, dt \text{ by symmetry} \]
\[ = \frac{ab}{2} \int_0^{2\pi} (1 + \cos 2t) \, dt \]
\[ = \frac{ab}{2} \left( t + \frac{\sin 2t}{2} \right) \bigg|_0^{2\pi} = \frac{ab}{2} (2\pi) = \pi ab \]

Note: if \( a = b = r \), we have \( a \) circle and \( A = \pi r^2 \)

Note: \( \int_0^{2\pi} \cos^2 t \, dt = \frac{\pi}{2} \)
\( \int_0^{2\pi} \sin^2 t \, dt = \pi \)