Solutions to Exam I (Blue)

1. Given the points $A(1, 2, 3)$, $B(1, 0, 1)$, and $C(2, 2, 2)$ find each of the following:

   (a) the length of $BC = \sqrt{(2 - 1)^2 + (2 - 0)^2 + (2 - 1)^2} = \sqrt{6}$

   (b) $\angle BAC \overrightarrow{AB} = \langle 0, -2, -2 \rangle$, $\overrightarrow{AC} = \langle 1, 0, -1 \rangle$, so

   \[
   \cos BAC = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \frac{0 + 0 + 2}{\sqrt{0 + 4 + 4\sqrt{1 + 0 + 1}} = \frac{1}{2}, \text{ so } \angle BAC = 60^\circ}
   \]

   (c) A vector orthogonal to the plane containing these points. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ lie in the plane, so $\overrightarrow{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 0 & -2 & -2 \\ 1 & 0 & -1 \end{vmatrix} = \langle 2, -2, 2 \rangle$.

2. Given the lines $\mathbf{r}(t) = \langle 2t, 2 - 3t, -2t - 1 \rangle$ and $x = 2s - 4$, $y = s - 2$, $z = 8 - 4s$:

   (a) Show the lines intersect. Find the point of intersection and the angle at which the lines intersect. \textbf{If they intersect, we must have} $2t = 2s - 4$, $2 - 3t = s - 2$, $-2t - 1 = 8 - 4s$. Solving the first 2 equations yields $t = \frac{1}{2}$, $s = \frac{5}{2}$, which also satisfy the third equation (MUST show!). Therefore, the intersection is $\mathbf{r}\left(\frac{1}{2}\right) = \langle 1, \frac{1}{2}, -2 \rangle$.

   (b) Find the equation of the plane containing these lines. The direction vectors lie in the plane, so $\overrightarrow{n} = \langle 2, -3, 2 \rangle \times \langle 2, 1, -4 \rangle = \begin{vmatrix} i & j & k \\ 2 & -3 & 2 \\ 1 & -4 \end{vmatrix} = \langle 14, 4, 8 \rangle$.

   Using either $\overrightarrow{n}$ or the intersection as our point, we obtain $14x + 4y + 8z = 0$.

3. (15 points) Given the surface $z = x^2 + 3y^2$:

   (a) Describe the traces in the planes $x = k, y = k, z = k$. $x = k$ yields parabolas, $y = k$ yields parabolas, and $z = k$ yields ellipses

   (b) Identify the surface (include name and any axes). An elliptic paraboloid along the $z$-axis.

   (c) Find the equation of the plane tangent to this surface when $x = 2, y = 1$. $\overrightarrow{n} = \langle -f_x, -f_y, 1 \rangle = \langle -2(2), -6(1), 1 \rangle$. When $x = 2, y = 1, z = 7$, so the equation of the plane is $-4x - 6y + z = -7$.

4. Given $\mathbf{r}(t) = \langle t, 2 \sin 2t, 4 \cos t \rangle$, find the unit tangent vector $\mathbf{T}(t)$ at the point where $t = \frac{\pi}{6}$. $\mathbf{r}'(t) = \langle 1, 4 \cos 2t, -4 \sin t \rangle$, so $\mathbf{r}'\left(\frac{\pi}{6}\right) = \langle 1, 2, -2 \rangle$. Therefore, $\mathbf{T}\left(\frac{\pi}{6}\right) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{-2}{3} \right\rangle$. 

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5. (Approximate answers acceptable) The radius and height of a cylindrical can are measured to be 60 cm and 75 cm respectively, with possible error of ± 0.5 cm for the radius and ± 0.2 cm for the height. Use differentials to estimate the largest possible error in calculating the surface area of the can. Recall
\[ SA = 2\pi r^2 + 2\pi rh. \]
\[ dSA = \frac{\partial SA}{\partial r} dr + \frac{\partial SA}{\partial h} dh = (4\pi r + 2\pi h)dr + (2\pi r)dh. \]
Substituting the above numbers yields
\[ dSA = 219\pi \text{cm}^2. \]

6. Given the surface \( g(x, y, z) = x \tan^{-1}\left(\frac{y}{z}\right) \):

(a) Find the directional derivative of \( g \) at the point \((1, 2, -2)\) in the direction of the vector \( \mathbf{i} + \mathbf{j} - \mathbf{k} \). \( \nabla g = \nabla g \cdot \mathbf{u} \).
\[ \nabla g = \left(\tan^{-1}\left(\frac{y}{z}\right), \frac{x}{1 + \left(\frac{y}{z}\right)^2}, \frac{x}{1 + \left(\frac{y}{z}\right)^2}\right) \left(\frac{-y}{z^2}\right) \]. Evaluate at the given point: \( \nabla g = \left(-\frac{\pi}{4}, -\frac{1}{4}, -\frac{1}{4}\right) \). \( \mathbf{u} \) is a unit vector in the direction of the given vector: \( \mathbf{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle \). Therefore,
\[ D_{\mathbf{u}} g = -\frac{\pi}{4\sqrt{3}} - \frac{1}{4\sqrt{3}} + \frac{1}{4\sqrt{3}} = -\frac{\pi}{4\sqrt{3}} \]

(b) Find the maximum rate of change in \( g \) at the point \((1, 2, -2)\) and the direction in which it occurs. The maximum rate of change is
\[ |\nabla g| = \sqrt{\frac{\pi^2}{16} + \frac{1}{16} + \frac{1}{16}} = \sqrt{\frac{\pi^2 + 2}{16}} \]
The direction it occurs is the direction of \( \nabla g = \left\langle -\frac{\pi}{4}, -\frac{1}{4}, -\frac{1}{4} \right\rangle \) (which was accepted in the text and therefore on the exam, or you make the unit vector \( \frac{\nabla g}{|\nabla g|} \))

7. Find and classify all critical points of the function \( f(x, y) = x^4 + y^4 - 4xy + 1 \) This is example 3 in the text (p776). The answer is \((0, 0)\) is a saddle point, \((1, 1)\) and \((-1, -1)\) are relative minima.