Take-home quiz #5: 12.7-12.8

Directions: Show all appropriate work in the space provided for each problem. Partial credit will be given for appropriate work. Calculators ARE allowed, but exact answers are required and sufficient analytic work must be shown to obtain full credit.

1. (5 pts) Find the absolute maximum and absolute minimum of $f(x, y) = y\sqrt{x} - y^2 - x + 6y$ on the square $[0, 9] \times [0, 5]$.
   $\nabla f = \left( \frac{1}{2} x^{-1/2} y - 1, \sqrt{x} - 2 y + 6 \right) = (0, 0)$ when $x = 4, y = 4$.
   Corner points are $(0, 0), (9, 0), (0, 5), (9, 5)$
   On the base of the square, $y = 0$, so $f(x, 0) = -x$ which has no critical values
   On the left of the square, $x = 0$, so $f(0, y) = -y^2 + 6y$, which has a critical value at $(0, 3)$
   At the top of the square, $y = 5$, so $f(x, 5) = 5\sqrt{x} - 25 - x + 30$, which has a critical value at $\left( \frac{25}{4}, 5 \right)$
   On the right of the square, $x = 9$, so $f(9, y) = 3y - y^2 - 9 + 6y$, which has a critical value at $\left( 9, \frac{9}{2} \right)$.
   Evaluating $f$ at all critical values yields a maximum of 12 at $(4, 4)$ and a minimum of $-9$ at $(9, 0)$.

2. (5 pts) A rectangular prism has one corner at the origin and the other corner in the first quadrant at a point on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Find the point which yields the largest possible volume.
   $V = f(x, y, z) = xyz, \nabla f = \lambda \nabla g; \ yz = \frac{2\lambda x}{a^2}, \ xz = \frac{2\lambda y}{b^2}, \ xy = \frac{2\lambda z}{c^2}.$ $x, y, z \neq 0$,
   so $\lambda = \frac{a^2 y z}{2 x} = \frac{b^2 x z}{2 y}$, $x = \frac{a}{b} y$. Similarly (using first and third equations)
   $z = \frac{c}{b} y$. Then $\frac{y^2}{b^2} + \frac{y^2}{b^2} + \frac{y^2}{b^2} = 1$, or $y = \frac{\pm b}{\sqrt{3}}, x = \frac{\pm a}{\sqrt{3}}, z = \frac{\pm c}{\sqrt{3}}$. Assuming $a, b, c$ positive, the volume is maximized at the point $\left( \frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}} \right)$.