Why Richardson’s Extrapolation?

We know at least three ways to approximate the first derivative.

\[
\begin{align*}
    f'(a) & \approx \frac{f(a + h) - f(a)}{h} & \text{Forward} \\
    f'(a) & \approx \frac{f(a) - f(a - h)}{h} & \text{Backward} \\
    f'(a) & \approx \frac{f(a + h) - f(a - h)}{2h} & \text{Symmetric}
\end{align*}
\]

From Taylor’s series we have for the forward expression that

\[
f(a + h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \cdots
\]

\[
\frac{f(a + h) - f(a)}{h} = f'(a) + \frac{1}{2}f''(a)h + \cdots
\]

We also have the backwards expression.

\[
f(a - h) = f(a) - f'(a)h + \frac{1}{2}f''(a)h^2 + \cdots
\]

\[
\frac{f(a - h) - f(a)}{-h} = f'(a) - \frac{1}{2}f''(a)h + \cdots
\]

Thus, the symmetric divided difference can be written as

\[
\frac{f(a + h) - f(a - h)}{2h} = f'(a) + \frac{2}{3!}f'''(a)h^2 + \frac{2}{5!}f^5(a)h^4 + \cdots
\]

It has a higher order of approximation being of order 2, whilst the other two are of order 1 (i.e. the power of \( h \)). The conclusion we can reach is that we have an exact formula for the derivative in terms of what we can compute (the left side) and the value we want to approximate together with an error formula. This gives the setting for Richardson’s extrapolation.

Richardson’s Extrapolation

As you have seen when subtracting almost equal quantities there is a serious loss of significance in the result. This is where Richardson’s extrapolation is at its best. We assume that the quantity we which to compute can be expressed as follows.

\[
V(h) = V + a_1h^2 + a_2h^4 + \cdots
\]

Keep in mind that we don’t know \( V \); this is what we want to approximate. Now computing \( V(h) \) and \( V(\frac{1}{2}h) \) gives...
\[ V(h) = V + a_1 h^2 + a_3 h^4 + \cdots \]
\[ V\left(\frac{1}{2} h\right) = V + a_1 \left(\frac{1}{2} h\right)^2 + a_2 \left(\frac{1}{2} h\right)^4 + \cdots \]
\[ V\left(\frac{1}{2} h\right) = V + \frac{1}{4} a_1 (h)^2 + \frac{1}{16} a_2 (h)^4 + \cdots \]

So, multiplying the \( V\left(\frac{1}{2} h\right) \) by four and subtracting from it \( V(h) \) there results

\[ 4V\left(\frac{1}{2} h\right) - V(h) = 3V + a'_2 h^4 + a'_3 h^6 + \cdots \]

Now divide by 3 to get

\[ \frac{1}{3} \left( 4V\left(\frac{1}{2} h\right) - V(h) \right) = V + a''_2 h^4 + a''_3 h^6 + \cdots \]

You can see that we now have a fourth order approximation. Call this approximation as

\[ V_2(h) = V + a''_2 h^4 + a''_3 h^6 + \cdots \]

The next time we will compute. Now repeat the process we began with. Compute the difference

\[ 16V_2\left(\frac{1}{2} h\right) - V_2(h) = 15V + a''' h^6 + \cdots \]

Now divide by 15, and we have racheted up the approximation another notch. Continue this process.

Can you see that are the multiples used at each stage?

The way this idea is used is to construct from the onset the values

\[ V(h), \ V\left(\frac{1}{2} h\right), \ V\left(\frac{1}{2^2} h\right), \ V\left(\frac{1}{2^3} h\right), \ldots \]

and compute all of the differences

\[ V_2\left(\frac{1}{2^{k-1}} h\right) = \frac{1}{3} \left( 4V\left(\frac{1}{2^k} h\right) - V\left(\frac{1}{2^{k-1}} h\right) \right) \]

and then

\[ V_3\left(\frac{1}{2^{k-1}} h\right) = \frac{1}{3} \left( 4V_2\left(\frac{1}{2^k} h\right) - V_2\left(\frac{1}{2^{k-1}} h\right) \right) \]

and so on. The order of the approximations become very, very good, very, very fast!