October 22, 2000

The Transition to Calculus[1]

1 Introduction

Let us say that the first half of the 17th century anticipated calculus while the second half realized calculus. Most readers of mathematical history are surprised by how much calculus was actually invented before Newton and Leibnitz took their turn. Let us note that by this time symbolism is well developed and its implications are well under development. With symbols, many, many expressions take on a compact form that reveal structure such as symmetries and formulas. The easiest formula to imagine is the exponent rule for powers. Moreover, the addition, multiplication and division of polynomials becomes transparently simple. We see at this time formal structures of proof, such as induction, are still missing — but applied nonetheless.

Countries such as France and the Netherland were wealthy and could afford the luxury of a large intellectual class from which many of our mathematicians of this era arose. The politics of discovery was very much unfettered, and with that mathematicians and scientists were free to pursue any idea of interest. By 1660, the stage was set for the grandest invention of our epoch, calculus and analysis.

2 Early Probability

$\star$ Early serious attempts at probability had already been attempted by Cardano and Tartaglia. They desired a better understanding of gambling odds. Some study about dice date even earlier. There are recorded attempts to understand odds dating back to Roman times.

Cardano published *Liber de Ludo Alea* (Book on Games of Chance) in 1526. He discusses dice as well stakes games. He then computes fair stakes based on the number of outcomes. He was also aware of independent events and the multiplication rule: if A and B are independent events then $p(A \text{ and } B) = p(A)p(B)$.

Cardano discussed this problem:

> How many throws must be allowed to provide even odds for attaining two sixes on a pair of dice?

Cardano reasoned it should be 18. He also argued that with a single dice, three rolls are required for even odds of rolling a 2. He was wrong. This type problem still challenges undergraduate math majors to this day.

The division of stakes problem was also of interest:

> A fair game ends when a player has won six rounds. Suppose that the game actually stops when the first player wins 5 games and the second play wins 3 games. How should the stakes be divided?

Tartaglia argued it should by $5:3$. This is incorrect. This work faded until about 1650 when **Blaise Pascal**(1623-1662) attempted it.

Through Roannez he met **Chevalier de Mere** who introduced him to games of chance and stimulated his interest in mathematical probability.Pascal described his solution to the de Méré division problem in letters to Fermat in 1654. His solution is contained in the following.

Theorem. Suppose the first player lacks r games winning the set while the second player lacks s games where both $r+s \geq 1$. If the game is interrupted at this point the first player gets

$$\sum_{k=0}^{s-1} \binom{n}{k} \Big/ 2^n,$$

where $n = r + s - 1$ (the maximum number of games left).

Example. Suppose $r = 1$ and $s = 3$, as in the Tartaglia example. Then the division (proportion) of stakes to Player 1 is

$$\sum_{k=0}^{3-1} \binom{n}{k} \Big/ 2^3 = \frac{7}{8},$$

not 5:3 as Tartaglia suggested. To prove this result Pascal used the **Pascal triangle** in a very clever way. In fact, he proves a great many theorems about the triangle.

Proof of Theorem. By induction on the maximum number of games to win, suppose that if Player 1 lacks r games to win and Player 2 lacks s games to win, then the division of stakes to Player 1 is in the ratio of

$$\sum_{k=0}^{s-1} \binom{n}{k} \quad \text{to} \quad 2^n$$

Now suppose there are $n+1$ games to play and a game is played. If Player 1 wins then there remain n games to play and $n = r-1+s-1$. If Player 2 wins then there remain n games to play and $n = r-s-1-1$. Since either event is equally likely, then the division of stakes is the average of the two divisions, which is

$$\sum_{k=0}^{s-1} \binom{n}{k} + \sum_{k=0}^{s-2} \binom{n}{k} \quad \text{to} \quad 2 \cdot 2^n.$$

This can be rewritten as

$$\binom{n}{0} + \sum_{k=1}^{s-1} \binom{n}{k} + \sum_{k=1}^{s-1} \binom{n}{k-1},$$

Now it is a property of "Pascal's" triangle that

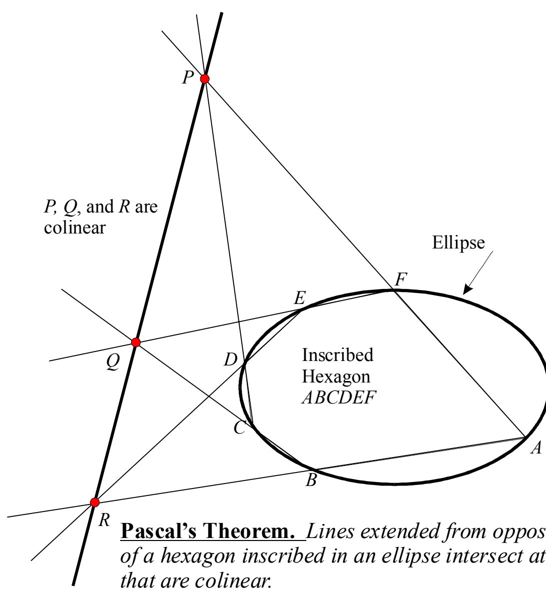
$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

from which the proof follows.

⋆ The first probability text (1657) was written by **Christian Huygens** (1629-1695), a student of Franz van Schooten (1615-1660). He covers thoroughly the division problem. In addition he gives the first proof of the other de Mere problem, that of the number of throws required for a pair of sixes to be fair. He concludes his book with a collection of probability problems concerning drawing colored balls from an urn. ⋆ Huygens was a distinguished mathematician and physicist in his own right. He studied the very important subject of optics and made significant contributions. ⋆ He also studied the cycloid and its relation to the pendulum clock.

3 Blaise Pascal

Blaise Pascal(1623-1662) was born of rich ancestry — merchants that attained the highest ranks of the burgess class. His father, Etienne, was a royal tax officer and sometime mathematician. Pascal appears to have had no formal education,having been "home-schooled" by his father. He continued his education in the salons and scientific gatherings he attended with his father.In 1640 Pascal wrote an essay on conics extending the work of Desargues in projective geometry, though it was never published. Only a few scholars like Leibniz and de la Hire saw the manuscript.



Pascal's Theorem. *Lines extended from opposite sides of a hexagon inscribed in an ellipse intersect at points that are colinear.*

Pascal began work on his calculating machine in 1642. For three years he worked to develop a working model. In 1649 he received a monopoly (from the crown) for maufacturing and producing his calculating machine. However, he probably made little from his invention.

The Pascal calculating machine

In 1646, he began his barometric experiments, continuing with them for 8 years. In 1654 he wrote *Traité de l'équilibre*, a short work devoted to the laws of hydrostatics and to the demonstration and description of the various effects of the weight of air. It was published posthumously.He then turned to his studies on arithmetic, combinatorial analysis and the calculus of probability as is is reflected in his correspondence with Fermat. Pascal wrote his *Traité du triangle arithmetique* in the same year but it was not distributed until 1665.

Pascal continued his work in mathematics with his *Éléments de géometrie* (1657), prepared upon the request of Arnauld. At the beginning of 1659 he devoted his energies to the perfecting of the theory of divisibles, which was a forerunner of the methods of integral calculus. He is said to have invented the hydraulic press.

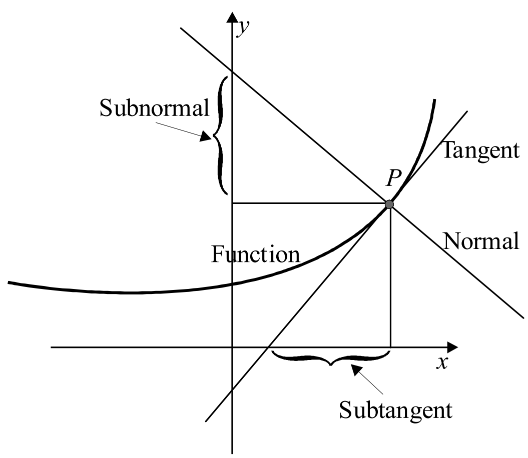
Pascal on a postage stamp

4 The Beginning of Calculus

As will become apparent a great many mathematicians experiment with the early concepts of calculus

- – Infinitesimal analysis/infinite series
- – Tangents and Normals
- – Subtangents and Subnormals
- – Areas and Volumes

Recall the picture.



We will see that "polynomial calculus", the calculus of infinite series, and even a little more was well worked over (as opposed to understood) long before **Newton** and **Lebnitz**. So it will be important to distinguish their monumental contributions from previous work. There are many issues to be considered, not least among them the

- concept of limit,
- the nature of the infinitesimal,
- and infinite decomposability of space.

First we examine original methods of constructing tangents and extrema, then areas and volumes, the intuitive power of power series, rectifiability. Then we discuss Newton and Leibnitz who organized it all into a comprehensive whole.

4.1 Fermat's Tangents and Extrema

Shortly after 1629 Fermat wrote the treatise *Method of Finding Maxima and Minima* in which he gives methods of

- Determining maxima and minima of *functions*
- Finding tangents to curves.

Finding extrema: In modern terms, Fermat would compare the value of a function $f(x)$ with the value of $f(x+E)$ at a neighboring point. Two situations arise:

- – At an ordinary point there is a distinct difference.
- – At an extrema the difference is imperceptible. Hence to find extrema Fermat would equate the two, divide by E, then set E to zero. In essence, Fermat is taking a limit

  $$\lim_{E \to 0} \frac{f(x+E) - f(x)}{E}.$$

  This process he calls **adequality**.

Example. Consider $f(x) = -x^2 + 3x - 2$. Then

$$\begin{aligned} f(x+E) &= -(x+E)^2 + 3(x+E) - 2 \\ &= -x^2 - 2xE - E^2 + 3x + 3E - 2. \end{aligned}$$

Now set $f(x) = f(x+E)$: subtracting gives

$$-2xE - E^2 + 3E = 0.$$

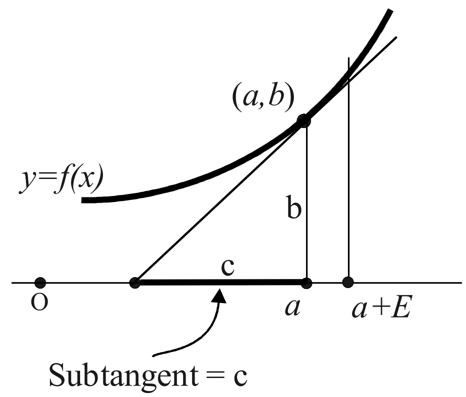
Divide by E:

$$-2x + 3 - E = 0.$$

Set $E = 0$.

$$\begin{aligned} -2x + 3 &= 0 \\ x &= \frac{3}{2}. \end{aligned}$$

This is the location of the maximum.



Finding Tangents: A similar idea is employed. The tangent at $f(a)$ $(b = f(a))$ very nearly includes the point $f(a+E)$. Therefore if c is the subtangent to $f(x)$ at (a,b)

$$\frac{b}{c} = \frac{f(a+E)}{c+E}.$$

This is the *adequality*. Cross multiply, cancel terms, divide by E, and finally set $E = 0$. This is equivalent to the usual difference quotient limiting procedure.

Example. Consider $f(x) = x^2 - 2x + 3$ and $a = 2$. Then $f(2) = 3$ Equate (adequality)

$$\frac{3}{c} = \frac{f(2+E)}{c+E}.$$

This gives

$$\begin{aligned} \frac{3}{c} &= \frac{(2+E)^2 - 2(2+E) + 3}{c+E} \\ &= \frac{4E + E^2 - 2E + 3}{c+E} \end{aligned}$$

Cross multiply to get

$$2cE + cE^2 + 3c = 3c + 3E.$$

Cancel, divide by E, and set $E = 0$ to get

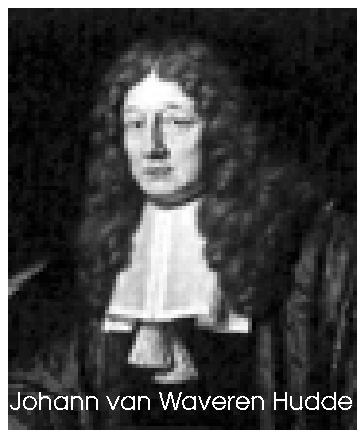
$$c = \frac{3}{2}.$$

The tangent is $b/c = 2$, which is indeed the tangent to $f(x)$ at $x = 2$.

Descartes' normals: Descartes' idea for finding a normal to a curve at a point P was to take a nearby point Q on the curve and find the equation of a circle with center on the coordinate axis (he used only abscissas). Now set to zero the discriminant of the equation that determines the intersections of the circle with the curve, one finds the center of the circle where Q coincides with P. The center being known, the tangent and normal are easily found. This method did not directly use infinitesimals and there was some hope that they would not be needed in further developments.

5 Hudde and Sluse

By the 1630's **Roberval** (1602-1675) had discovered kinematic methods of determining tangents by considering a curve to be generated by a moving point. His method relied on geometry and was not algorithmic. The methods of Fermat and Descartes could involve complicated algebra. However, **Johann Hudde** (1628-1704) and **René François de Sluse** (1622-1685) found easily used algorithms. These algorithms were to be adapted by Newton.


Johann van Waveren Hudde

Hudde established two results:

1. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has a double zero at $x = \alpha$, and if $p_0, p_1, \ldots$ is an arithmetic progression, then $x = \alpha$ is a root of
$$q(x) = a_n p_n x^n + a_{n-1}p_{n-1}x^{n-1} + \cdots + a_0 p_0.$$

2. If at $x = \alpha$ the polynomial
$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$
has a relative maximum or minimum, then
$$na_n x^n + (n-1)a_{n-1}x^{n-1} + \cdots + a_1 x = 0.$$

(Note, this equals $xp'(x)$.)

It should be apparent that not only did area and tangent problems pre-date the calculus of Newton and Leibnitz but so did many of the rules for derivatives!

de Sluse established the following result for the curve $f(x, y) = 0$, where $f(x, y)$ is a polynomial: In modern terms it reads:

$$\text{subtangent for } f(x) = -\frac{y f_y(x, y)}{f_x(x, y)}.$$

The proof follows: For $f(x, y) = 0$ we have

$$f_x + f_y y' = 0.$$

So

$$y' = -\frac{f_x(x, y)}{f_y(x, y)}.$$

Now $y' = y/t$, where t is the subtangent and thus

$$t = -\frac{y f_y(x, y)}{f_x(x, y)}.$$

Sluse gives nothing like a proof, not even a clue to discovery. He may have just generalized numerous examples.

Example. We apply the algorithm as Sluse gave it for the subtangent of $f(x, y) = x^5 + bx^4 - 2q^2y^3 + x^2y^3 - b^2 = 0$. First remove all constant terms. Then move all the x terms to the left side of the equation and the y terms to the right. Terms with both x and y terms are placed on both sides, those on right side multiplied by a negative one. So,

$$x^5 + bx^4 + x^2y^3 = 2q^2y^3 - x^2y^3.$$

Note the x^2y^3 term!! Now multiply all terms on the right by its y exponent and all terms on the left by its x exponent. This gives

$$5x^5 + 4bx^4 + 2x^2y^3 = 6q^2y^3 - 3x^2y^3$$

Next, one the left side replace one of the powers of x by t in each term.

$$5x^4t + 4bx^3t + 2xy^3t = 6q^2y^3 - 3x^2y^3$$

Solve for t, the subtangent,

$$t = \frac{6q^2y^3 - 3x^2y^3}{5x^4 + 4bx^2 + 2xy^3},$$

which is the subtangent.

What is significant is that this is an algorithm that may be applied to any polynomial curve, with no special-to-the-curve methods required. Now anyone can compute tangents!!!!

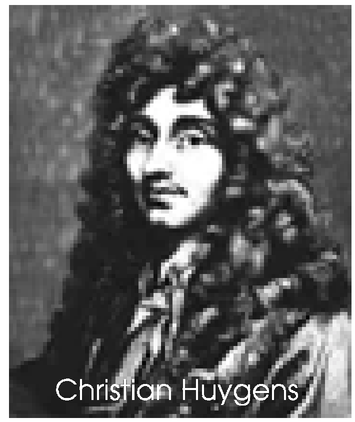
René François de Sluse, a Belgian, came from a petit noble family. His father was a lawyer, notary and a clerk of the court. He attended the University of Louvain from 1638 to 1642. He then studied in Rome, receiving his doctorate in law from the University of Sapienza in 1643. He remained in Rome for several more years becoming proficient in Greek, Hebrew, Arabic, and Syriac, and studying astronomy and mathematics.

Sluse made a thorough study of Cavalieri and Torricelli on the geometry of the indivisible. At Liège, because of administrative duties, his only means of communication was his extensive correspondence. In order to keep his scientific interests alive he conducted extensive correspondence with several members of the scientific community, Pascal, Huygens, Oldenburg, Wallis, Ricci, Dati, Lambecius, and Prince Leopold of Tuscany.

Early in his career he published *Mesolabium*, a work on geometrical construction in which he discussed the cubature of various solids and the solutions to third and fourth degree equations. He perfected the methods of Descartes and Fermat for drawing tangents and determining the maximum and minimum values. He generalized the method for solutions of equations through the construction of roots by means of curves. Sluse also wrote on astronomy, physics, natural history, history, and of course on theological issues in his adminstrative work.

6 Christiaan Huygens

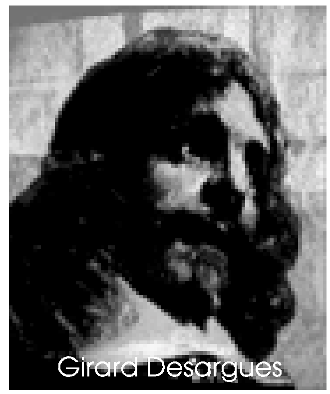
Christiaan Huygens (1629-1695)of the Netherlands was manifestly wealthy. From 1637 to 1643, he was educated privately at home by father and private tutors. 1644, tuition in mathematics by Stampioen. From 1645-1647, he studied law and mathematics (privately with Frans Van Schooten), University of Leiden. In 1655, he invented pendulum clock for the purpose of longitude determination, and sent them on sea trials (1662 & 1686). In 1664 (Dutch) and 1665 (French), he obtained patents for longitude clock.



He developed a diaphragm placed inside a telescope near the focus and used it with a pendulum that counted time to measure celestial angles. He also inserted a wedge shaped piece at the focus to cover a planet, in effect a micrometer. Corresponded with Mersenne, Gregory of St. Vincent, Wallis, Van Shooten, Sluse, Leibnitz, Roemer, Pascal, Fermat, Boulliau, and Oldenburg. He also mentored Leibnitz, when Leibnitz was just beginning to "think about mathematics". He generously offered Leibnitz essential advise.

7 Gérard Desargues

The Frenchman Gérard Desargues(1591-1661) had no university education. However, he was a profound thinker, primarily in geometry. His geometrical works marked an important step in the rationalization of graphical techniques. He introduced the principal concepts of projective geometry, trying to integrate the projective methods into the body of mathematics. However, his work was rediscovered and fully appreciated only in the 19th century.



His works were collected in *L'oeuvre mathématique de Desarques* (Paris, 1951). As an engineer, he designed many projects. One of them was a system for raising water near Paris. The system was based on the relatively unknown epicycloidal wheel. He became friendly with Mersenne, Gassendi, Mydorge, and perhaps Roberval. In 1635 he attended the meetings of Mersenne's Académie parisienne. In 1638 he had contacts with Descartes and in 1639 with Pascal.

8 Albert Girard

Of the Frenchman Albert Girard (1595-1632) our first reliable information about him shows that he settled in the Netherlands, where it is known that he studied in Leiden (∼1617). He was a Calvinist

- He published extensively on mathematics.
- He worked on the law of refraction.
- Originally he was a musician, specifically a professional lute player.
- His grave marker called him an engineer.
- He appears to have had informal contact with the circle of Dutch mathematical scientists, thus his edition of Stevin. He was a friend of Snell (of Snell's law).

He was primarily an algebraist that conjecture the fundamental theorem of algebra, which is that every polynomial has at least one root.

9 Marin Mersenne

Marin Mersenne (1588-1648) entered the new Jesuit college at La Fleche fro 1604 to 1609. He then studied theology for two years at the Sorbonne, though there is no evidence of a degree. (He was from a family of laborers.)

Mersenne is best known for his network of correspondents. In addition to his many letters, Mersenne wrote several works on varying topics. His *Quaestiones in Genesim* (1623) defended orthodox theology against deists and atheists. In response to the work of Galileo, Mersenne wrote his *Traité des mouvements* (1633) and *Les mechaniques de Galilée* (1634). His *Traité d'harmonie universelle* (1627) was a work on music, acoustics and instruments which he continued to improve throughout his life.

From 1623 Mersenne began to make the careful selection of scholars who met at his convent in Paris or corresponded with him from all over Europe and as far afield as Tunisia, Syria, and Constantinople. His regular visitors or correspondents came to include Descartes, the Roman musicologist Giovanni Battista Doni, Roberval, Fermat, Hobbes, and the Pascals (father and son). It was in Mersenne's salon that the young Blaise Pascal met Descartes, 1647. His role as secretary of the public of scientific letters, with a strong point of view of his own, became institutionalized in the Academia Parisiensis, which he organized in 1635.