RESEARCH STATEMENT

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My research interest has been to develop and analyze effective numerical methods for multiscale phenomena. The main purpose is to develop innovative numerical methods that can capture the effect of small scales on the large scales without resolving the small scale details on a coarse computational grid. This research activity is strongly motivated by many important practical applications, ranging from porous media simulations to turbulent flow. During my graduate study at Texas A&M University, I have been working on designing and analysis of multiscale methods for (a) turbulent diffusion (b) high contrast parabolic equations (c) time-dependent nonlinear parabolic equations. I have also worked on application of multiscale simulation models for uncertainty quantification of Richards' equation.

1.1. **Turbulent diffusion transport governed by cellular flows.** The goal is to develop an efficient multiscale numerical technique for solving turbulent diffusion equation. We consider the steady convection-diffusion problem

$$\begin{aligned} &\varepsilon \Delta T^{\varepsilon} - u \cdot \nabla T^{\varepsilon} = 0 \quad \text{in} \quad \Omega, \\ &T^{\varepsilon}(x) = T_0(x) \quad \text{on} \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a simply connected bounded domain in R^2, the flow u is incompressible: $\nabla \cdot u = 0$, and the small parameter $\varepsilon > 0$ is the reciprocal of the Péclet number. Due to the incompressibility of u there exists a stream function $H(x, y)$ such that $u = (-H_y, H_x)$.

It was shown in [6] that the oscillation of the solution in the regions which are away from the separatrices by $O(\sqrt{\varepsilon})$ in each cell is small, and the solution near the separatrices can be approximated by an asymptotic problem on the graph of the separatrices. Furthermore, the solution inside each cell converges to a constant. It was also shown that as $\varepsilon \to 0$ the asymptotic Childress problem has a unique periodic solution which is obtained by solving a system of one-dimensional heat equations on the graph in the boundary layer coordinates. The periodic solution tends to a constant at infinity which is obtained by imposing the Neumann conditions at infinity.

In order to solve the system of heat equations, we use spectral methods and finite difference methods on exponential grids. Spectral methods, e.g. Galerkin approach [3, 7] and collocation method, are considered to solve heat equations over the unbounded domain. The Laguerre and Hermite functions are used as basis elements in the Galerkin formulation. Although the computational cost of the spectral approach is usually inexpensive and fast, the main difficulty is determining the constant states. For each cell, the constant to which the solution inside each cell converges is unknown, and our goal is match all the solutions of cells after constructing the solution of each cell. Instead, finite difference methods on the non-uniform exponential grids are performed to capture the constant for each cell. Indeed, the Laguerre and Hermite functions decay exponentially and it is expensive to use finite difference scheme with uniform grid points over unbounded domain. Thus, we develop the finite difference approach on the exponential grids. We have solved this problem for 1-cell case, 2-cell case, and 4-cell case and, currently, are working on generalization to multiple-cell case by minimizing unknowns [4].

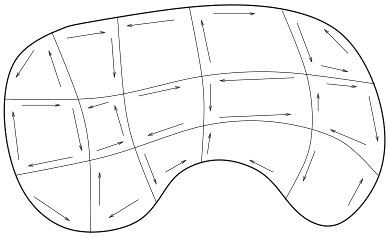

FIGURE 1. The domain partitioned by flow separatrices into cells

1.2. **High contrast parabolic equations.** A modified multiscale finite element method for the computations of two-phase flows was proposed in [1]. The main idea of this approach is to use a global information in constructing finite element basis functions. This approach is intended for the problems without apparent scale separation. We analyze this approach for the modified multiscale finite element methods for parabolic equations in strongly channelized media. We consider the parabolic equations

$$D_t p = \nabla \cdot (\kappa(x,t) a_\varepsilon(x) \nabla p) \quad \text{in } \Omega, \tag{2}$$

where Ω is a unit square in R^2. Let us denote by p_0 the steady solution of the parabolic equation

$$\nabla \cdot (a_\varepsilon(x) \nabla p_0) = 0,$$

with the same boundary condition as in (2). Denote by $\psi_0 = \psi(t = \infty)$ the stream function at steady state defined by zero at the bottom edge. For the analysis, we use streamline-pressure coordinates $\eta = \psi_0, \zeta = p_0$. Under some assumptions, we justify rigorously the formal asymptotic expansion

$$p(\eta, \zeta, t) = p_0(\zeta, t) + \delta p_1(\eta, \zeta, t) + \cdots ,$$

which implies that the pressure mainly depends on ζ. Currently, using this asymptotic expansion we are analyzing the modified multiscale finite element methods [2].

1.3. **Time-dependent parabolic equations.** I have studied numerical homogenization techniques for nonlinear parabolic problems.

$$D_t u_\varepsilon - div(a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u_\varepsilon, D_x u_\varepsilon)) + a_0(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u_\varepsilon, Du_\varepsilon) = f. \tag{3}$$

A numerical homogenization procedure of nonlinear parabolic equations is proposed and analyzed in [5]. The numerical homogenization procedure is the following. For each

$\eta \in R$, we solve

$$\begin{aligned} D_t v_\varepsilon - div(a(x/\varepsilon, t/\varepsilon^\alpha, \eta, D_x v_\varepsilon)) &= 0 \text{ in } K \times (t_n, t_{n+1}), \\ v_\varepsilon(t = t_n) &= v_\varepsilon^h, \\ v_\varepsilon\,|_{\partial K} &= v_\varepsilon^h, \end{aligned} \tag{4}$$

where v_ε^h is the linear function. Then the homogenized flux is computed by

$$a^*(\eta, D_x v_\varepsilon^h) = \frac{1}{|K|} \int_{t_n}^{t_{n+1}} \int_K a(x/\varepsilon, t/\varepsilon^\alpha, \eta, D_x v_\varepsilon) dx dt.$$

The goal of this procedure is that the solution of the homogenized equation can be approximated by the solution of the local problems (4). The procedure is defined in the following way. Find $u_h(t) \in S_h$ such that

$$\int_{t_n}^{t_{n+1}} \int_\Omega D_t u_h v_h dx dt + A(u_h, v_h) = \int_{t_n}^{t_{n+1}} \int_\Omega f v_h dx dt \quad \text{for all } v_h \in S^h,$$

where

$$\begin{aligned} A(u_h, v_h) \;=\; \sum_K \int_{t_n}^{t_{n+1}} \int_K &((a(x/\varepsilon, t/\varepsilon^\alpha, \eta^{u_h}, D_x u_{h,\varepsilon}), D_x w_h) \\ &+ a_0(x/\varepsilon, t/\varepsilon^\alpha, \eta^{u_h}, D_x u_{h,\varepsilon}) w_h) dx dt, \end{aligned}$$

where $u_{h,\varepsilon}$ is the solution of the local problem (4), $\eta^{u_h} = \frac{1}{|K|} \int_K u_h dx$, u_h is known at $t = t_n$, and S_h is a standard finite dimensional space over a coarse triangulation of Ω. The numerical homogenization procedure can be formulated within the framework of multiscale finite element methods which was capable of correctly capturing the oscillations of the solution as well as finding the coarse scaled solution.

Our goal is to obtain the explicit convergence rates for the numerical homogenization procedure. In particular, we would like to estimate $\|u - u_h\|_{L^p(0,T;W_0^{1,p}(\Omega))}$, where u is a solution of the homogenized equation. The main difficulty is to obtain sharp estimates for correctors. The correctors are defined as follows:

$$P = D_x v_\varepsilon^h + \varepsilon D_x N_{\eta^h, D_x v_\varepsilon^h}(x/\varepsilon),$$

where $N_{\eta^h, D_x v_\varepsilon^h}$ is a zero mean periodic function satisfying the following:

$$D_\tau N - div(a(y, \tau, \eta^h, D_x v_\varepsilon^h + D_y N)) = 0 \;\text{ in } K.$$

Once the estimate for the correctors is found, we can obtain the convergence of the numerical homogenization using comparison principles, compactness arguments, and discrete Meyers type estimates. Note that the numerical homogenization provides us an approximation for the homogenized solution. To obtain the approximation for oscillatory solutions, we introduce numerical correctors and find the estimates for the convergence of the numerical correctors.

1.4. **Applications of coarse-scale models to uncertainty quantification in Richards' equation.** I have studied efficient uncertainty quantification techniques for Richards' equation which use coarse-scale simulation models developed in earlier section. Richards' equation describes the infiltration of water into a porous media whose pore space is filled with air and water. Our problems are motivated by applications to soil moisture predictions. Soil moisture conditions are important in determining the amount of infiltration and ground water recharge. Soil moisture is controlled by factors such as soil type, topography, vegetation, and climate. Soil moisture is typically measured at different scales varying from point scale (*in-situ*) to remote sensing scale (of order several kilometers). The objective is to predict the soil moisture at different resolutions via prediction of the saturated hydraulic conductivity field.

In this work, we employ Markov chain Monte Carlo (MCMC) methods for sampling the saturated hydraulic conductivity field given average dynamic data (e.g., average flux). Direct MCMC methods are very expensive because each iteration requires the computation with fine-scale simulations. We propose the use of coarse-scale models in this sampling problem with a goal to reduce the computational cost. I am currently working on this project. Our initial results show that one can improve the computational cost of sampling procedure by using coarse-scale models. This is a joint work with the group in Agricultural and Biological Engineering department.

References

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