Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

# MEYERS TYPE ESTIMATES FOR APPROXIMATE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS AND THEIR APPLICATIONS

Yalchin Efendiev

Department of Mathematics
Texas A & M University
College Station, TX 77843-3368

Alexander Pankov

Department of Mathematics
College of William & Mary
Williamsburg, VA 23187-8795

Abstract. In this paper we obtain Meyers type regularity estimates for approximate solutions of nonlinear elliptic equations. These estimates are used in the analysis of a numerical scheme obtained from a numerical homogenization of nonlinear elliptic equations. Numerical homogenization of nonlinear elliptic equations results in discretization schemes that require additional integrability of the approximate solutions. The latter motivates our work.

## 1. **Introduction.**

Meyers type regularity estimates for nonlinear differential equations have been known and used for some time [12]. In this paper our goal is to derive such estimates for approximate solutions without using the solutions of the continuous equations. The need for such estimates arises in different situations. Our interest in these estimates stems from the numerical homogenization of nonlinear elliptic equations [4, 3]. Within this procedure, after homogenizing over the spatial heterogeneities, one obtains a discrete equation that is not a standard Galerkin discretization of the original equation. To analyze the convergence of the method, one needs Meyers type estimates for discrete solutions.

In this paper our goal is to obtain Meyers type regularity estimates for approximate solutions of general elliptic equations. We apply the techniques presented in [5] for continuous equations to discrete problems. This technique goes back to [8] (cf. [9]). We would like also to mention the paper [6] where nonstandard Meyers type estimates are obtained. The starting point for this approach is the use of regularity estimates for linear (Laplace) equations. Further, employing the linear operators we introduce the contraction maps that allow us to obtain Meyers type estimates. We derive these estimates first for strongly monotone operators. Furthermore, using a particular discrete solution we obtain Meyers type estimates for more general elliptic operators of the form

$$-div(a(x,u,Du)) + a_0(x,u,Du) = f \tag{1}$$

2000 *Mathematics Subject Classification.* Primary: 35J60, 65N30; Secondary: 35J99 .
*Key words and phrases.* elliptic, nonlinear, finite element.

with homogeneous Dirichlet boundary conditions. It seems the same approach can be extended to the case of Neumann and mixed boundary conditions (see [5] for corresponding continuous problems). To obtain Meyers type estimates for (1), we use weaker assumptions than those imposed in [5].

In this paper we also consider an application of these estimates to a particular discretization of (1), which arises in numerical homogenization of such equations, [3, 4]. Numerical homogenization method is used for (1) with multiscale coefficients and allow us to compute the homogenized (averaged) solutions on a coarse grid. The discretization of (1) that we are interested in is different from the standard Galerkin discretization of this equation. In particular, two different discrete spaces are involved for approximation of $u$ and $Du$ in the fluxes. This scenario cannot be avoided in a numerical homogenization procedure because the solution of the local problems do not belong to the discrete spaces that are used for approximation of the homogenized solutions [4]. To obtain the convergence of the discrete solutions to a solution of (1), one needs Meyers type estimates. Our convergence result does not contain *apriori* error estimates because we impose general assumptions on the fluxes. The *apriori* convergence rates can be obtained under additional assumptions. However, in numerical homogenization *apriori* error estimates cannot be obtained for general heterogeneities (see [4] for further discussion). One of the reasons is that in the limit as the physical scale approaches to zero, there is no explicit convergence rates in homogenization for problems with general heterogeneities, where the fluxes can be discontinuous functions of spatial variables. For this reason, we are interested only in the convergence of approximate solutions in this paper.

The paper is organized as follows. In the next section, we discuss preliminary results regarding linear equations. In the following section, we obtain the estimates for monotone operators. Section 4 is devoted to the Meyers type estimates for equations (1). Finally, we use these estimates to prove the convergence of a numerical scheme.

2. **Preliminaries.** Let $E_h$ be a family of finite dimensional subspaces in $W_0^{1,p}(Q)$, $2 \le p \le p_0$ (some fixed $p_0$), and $Q$ is an open bounded domain in $\mathbb{R}^n$ with Lipschitz boundary. Consider

$$-\Delta u = f,$$

$u \in W_0^{1,2}(Q)$. For any $f \in W^{-1,2}(Q)$ there exists a unique solution $u \in W_0^{1,2}(Q)$, such that

$$\|u\|_{W_0^{1,2}(Q)} \le \|f\|_{W^{-1,2}(Q)}.$$

Moreover, if $f \in W^{-1,p}(Q)$, $2 \le p \le p_0$, then [5] $u \in W_0^{1,p}(Q)$ and

$$\|u\|_{W_0^{1,p}(Q)} \le C\|f\|_{W^{-1,p}(Q)}.$$

Moreover, if the domain is $C^1$, this holds true for all $p \ge 2$ [16]. Here, the norms in Sobolev spaces are defined as follows ($Du = grad\ u$):

$$\|u\|_{W_0^{1,p}(Q)} = \|Du\|_{L^p(Q)},$$

$\|\cdot\|_{W^{-1,p}(Q)}$ is the dual norm to $\|\cdot\|_{W_0^{1,q}(Q)}$, $1/p + 1/q = 1$, i.e.,

$$\|f\|_{W^{-1,p}(Q)} = \sup_{v \in W_0^{1,q}(Q)} \frac{|(f,v)|}{\|v\|_{W_0^{1,q}(Q)}}.$$

Here, following common practice, $(u,v)$ means $\int_Q uvdx$ and the duality pairing between $W^{-1,p}$ and $W^{1,q}$.

Note that $D : W_0^{1,p}(Q) \to L^p(Q)^n$ is an isometric imbedding. By duality, $div : L^p(Q)^n \to W^{-1,p}(Q)$ is onto and $\|div\, u\|_{W^{-1,p}(Q)} \le \|u\|_{L^p(Q)}$.

Consider approximate solutions of $-\Delta u = f$, i.e.,

$$(Du, Dv) = (f, v) \ \forall\ v \in W_0^{1,2}(Q).$$

The approximate problem is to find $u_h \in E_h$ such that

$$(Du_h, Dv_h) = (f, v_h),\ \forall v_h \in E_h. \tag{2}$$

There exists a unique solution $u_h \in E_h$. Our basic assumption is that for any $f \in W^{-1,p}(Q)$, $2 \le p \le p_0$, we have

$$\|u_h\|_{W_0^{1,p}(Q)} \le C_p \|f\|_{W^{-1,p}(Q)}. \tag{3}$$

When $Q$ is a polygon or a polyhedron, assumption (3) holds for some finite elements (see Theorem 7.5.3 of [2]). One can formulate conditions for finite element spaces that would guarantee (3) (see pages 170-171, [2]). These conditions hold for all the elements studied in Chapter 3 of [2]. In particular, quasi-uniform mesh is assumed. We would like to note that, our results hold when (3) is satisfied. The verification of (3) can be considered as a separate problem.

Assumption (3) is equivalent to the following one. Let $P_h$ be the orthogonal projection in $W_0^{1,2}(Q)$ onto $E_h$. Then

$$\|P_h u\|_{W_0^{1,p}(Q)} \le C_p \|u\|_{W_0^{1,p}(Q)}$$

for any $u \in W_0^{1,p}(Q)$. Note that one can take $C_2 = 1$.

Next, we would like to introduce the best possible $C_p$. Denote by $L_h$ the linear operator that maps $f$ into $u_h$,

$$L_h f = u_h.$$

Inequality (3) implies that $L_h$ is a bounded linear operator from $W^{-1,p}(Q)$ into $W_0^{1,p}(Q)$. Denote by $M_{p,h}$ the norm of the operators $L_h$ and set

$$M_p = \sup_h M_{p,h}.$$

Clearly, we have $M_2 = M_{2,h} = 1$.

Further, we introduce another family of operators $B_h$,

$$B_h = D \circ L_h \circ div.$$

The operator $B_h$ is a bounded linear operator that acts in the space $L^p(Q)^n$ and its norm in this space is equal to the norm of

$$L_h : W^{-1,p}(Q) \to W^{1,p}(Q).$$

Indeed,

$$\begin{aligned} \|B_h\| = \sup_{\|u\|_{L^p(Q)} \le 1} \|D \circ L_h \circ div u\|_{L^p(Q)} = \sup_{\|f\|_{W^{-1,p}(Q)} \le 1} \|D \circ L_h \circ f\|_{L^p(Q)} = \\ \sup_{\|f\|_{W^{-1,p}(Q)} \le 1} \|L_h \circ f\|_{W_0^{1,p}(Q)} = \|L_h\|. \end{aligned} \tag{4}$$

Next, we apply the Riesz-Thorin interpolation theorem [1] on $B_h$. Let $s > 2$, $2 \le p \le s$ and

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{s}.$$

Then using Riesz-Thorin interpolation theorem, we have

$$M_{p,h} \leq (M_{s,h})^{\theta}. \tag{5}$$

Taking the supremum with respect to $h$ we obtain

$$M_p \leq (M_s)^{\theta}. \tag{6}$$

Note that $M_p$ continuously depends on $p$.

3. **Motivation.** As we mentioned in Introduction, our main motivation in deriving Meyers type estimates for discrete solutions stems from numerical homogenization applications. Numerical homogenization of nonlinear elliptic equations results in discretization schemes that require additional integrability of the approximate solutions. In particular, we consider $u_\epsilon \in W_0^{1,p}(Q)$

$$-div(a_\epsilon(x, u_\epsilon, Du_\epsilon)) + a_{0,\epsilon}(x, u_\epsilon, Du_\epsilon) = f, \tag{7}$$

where $a_\epsilon$ and $a_{0,\epsilon}$ satisfy the conditions formulated in Section 4. Many nonlinear transport processes are described with this type of equations. Our interest is in the applications arisen in the steady state Richards's equation in heterogeneous soils [14], nonlinear convection-diffusion in heterogeneous media [3], the transport of two-phase immiscible flow, and, in general, the transport of multi-phase multi-component flows [7]. The homogenization of (7) is studied in [13]. Typical numerical homogenization procedures compute the effective fluxes on the coarse-grid (coarse-grid is a grid whose size is much larger than $\epsilon$) and the resulting equations are solved on this coarse-grid. Next, we briefly mention the numerical homogenization procedure. Consider a finite dimensional space over the standard triangular partitions $K$ of $Q = \bigcup K$, and let $S^h = \{v_h \in C^0(\overline{Q})$ : the restriction $v_h$ is linear for each element $K$ and $v_h = 0$ on $\partial Q\}$, $diam(K) \leq Ch$. Here we assume that $h \gg \epsilon$ is chosen for the approximation of the homogenized solution. The numerical homogenization procedure consists of finding an approximation, $u_h \in S^h$, of a homogenized solution $u$ such that

$$(A_{\epsilon,h} u_h, v_h) = \int_Q f v_h dx, \ \forall v_h \in S^h, \tag{8}$$

where

$$(A_{\epsilon,h} u_h, v_h) = \sum_K \int_K ((a_\epsilon(x, \eta^{u_h}, Du_{\epsilon,h}), Dv_h) + a_{0,\epsilon}(x, \eta^{u_h}, Du_{\epsilon,h}) v_h) dx. \tag{9}$$

Here $u_{\epsilon,h}$ satisfies

$$-div(a_\epsilon(x, \eta^{u_h}, Du_{\epsilon,h})) = 0 \text{ in } K, \tag{10}$$

$u_{\epsilon,h} = u_h$ on $\partial K$ and $\eta^{u_h} = \frac{1}{|K|}\int_K u_h dx$ in each $K$. Our numerical homogenization procedure consists of (8), (9) and (10). In some sense (9) attempts to approximate $\int_Q [(a^*(x, u_h, Du_h), Dv_h) + a_0^*(x, u_h, Du_h) v_h] dx$ which is a finite element formulation of the homogenized equation. However, in the limit as $\epsilon \to 0$, (8) becomes (see [3])

$$\int_Q [(a^*(x, \eta^{u_h}, Du_h), Dv_h) + a_0^*(x, \eta^{u_h}, Du_h) v_h] dx = \int_Q f v_h dx, \ \forall v_h \in S^h,$$

which is different from the standard finite element discretization. To prove $u_h \to u$ in an appropriate sense, we need Meyers type estimates. One can slightly change the variational formulation (8), however, the Galerkin discretization of the homogenized equation can not be obtained, in the limit $\epsilon \to 0$ and Meyers type estimates are required to prove the convergence of discrete solutions.

4. **Monotone operators.** Consider

$$Au = -div(a(x, Du))$$

and assume that

- $a : Q \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function and $a(x, 0) = 0$ (for the sake of simplicity).
- $$|a(x, \xi_1) - a(x, \xi_2)| \leq M|\xi_1 - \xi_2| \tag{11}$$
  for all $x \in Q$ and $\xi_1, \xi_2 \in \mathbb{R}^n$.
- $$(a(x, \xi_1) - a(x, \xi_2)) \cdot (\xi_1 - \xi_2) \geq m|\xi_1 - \xi_2|^2. \tag{12}$$

Here $|\cdot|$ stands for the Euclidean norm in $R^n$.

$A$ is a strictly monotone continuous operator from $W_0^{1,2}(Q)$ to $W^{-1,2}(Q)$, and

$$(Au - Av, u - v) \geq m\|u - v\|^2_{W_0^{1,2}(Q)}.$$

Hence, it is coercive. In fact, $A$ maps $W_0^{1,2}(Q)$ continuously into $W^{-1,2}(Q)$,

$$\|Au - Av\|_{W^{-1,2}(Q)} \leq M\|u - v||_{W_0^{1,2}(Q)}.$$

Consider the following approximate problem. Find $u_h \in E_h$ such that

$$(Au_h, v_h) = (f, v_h),\ \forall v_h \in E_h. \tag{13}$$

Our goal is to prove the following theorem.

**Theorem 1.**

$$\|u_h\|_{W_0^{1,p}(Q)} \leq \frac{tM_p}{1 - M_p k}\|f\|_{W^{-1,p}(Q)}, \tag{14}$$

*where*

$$t = \frac{m}{M^2}, \quad k = (1 - \frac{m^2}{M^2})^{1/2}.$$

To prove this theorem, we will need to reformulate the approximation problem (13) and study its properties. The proof of the theorem is presented at the end of this section. First note that

$$L_h(-\Delta u_h) = u_h,\ \forall u_h \in E_h. \tag{15}$$

Indeed, $(-\Delta u_h, v_h) = (Du_h, Dv_h)$, $\forall v_h \in E_h$ implies $u_h = L_h(-\Delta u_h)$. Consider the operator $\overline{A}$ defined by

$$\overline{A}u = -\Delta u - tA(u).$$

**Lemma 1.**

$$\|\overline{A}u_1 - \overline{A}u_2\|_{W^{-1,p}(Q)} \leq k\|u_1 - u_2\|_{W_0^{1,p}(Q)},$$

*i.e.,* $\overline{A} : W_0^{1,p}(Q) \to W^{-1,p}(Q)$ *is Lipschitz continuous with Lipschitz constant* $k$, $k < 1$.

**Proof.** The flux corresponding to $\overline{A}$ is given by

$$\overline{a}(x, \xi) = \xi - ta(x, \xi).$$

Next, we would like to derive the following estimate for $\overline{a}(x, \xi)$,

$$|\overline{a}(x, \xi_1) - \overline{a}(x, \xi_2)| \leq (1 - \frac{m^2}{M^2})^{1/2}|\xi_1 - \xi_2| = k|\xi_1 - \xi_2|. \tag{16}$$

Indeed,

$$|\overline{a}(x,\xi_1)-\overline{a}(x,\xi_2)|^2=(\xi_1-\xi_2)\cdot(\xi_1-\xi_2)-2t(a(x,\xi_1)-a(x,\xi_2))\cdot(\xi_1-\xi_2)+ t^2(a(x,\xi_1)-a(x,\xi_2))\cdot(a(x,\xi_1)-a(x,\xi_2))= |\xi_1-\xi_2|^2-2t(a(x,\xi_1)-a(x,\xi_2))\cdot(\xi_1-\xi_2)+t^2|a(x,\xi_1)-a(x,\xi_2)|^2. \tag{17}$$

Assumptions (11) and (12) imply

$$|\overline{a}(x,\xi_1)-\overline{a}(x,\xi_2)|^2\le|\xi_1-\xi_2|^2-2tm|\xi_1-\xi_2|^2+t^2M^2|\xi_1-\xi_2|^2= (1-2tm+t^2M^2)|\xi_1-\xi_2|^2=k^2|\xi_1-\xi_2|^2. \tag{18}$$

The estimate (16) implies immediately that for any $u_1,u_2\in W_0^{1,p}(Q)$ and $v\in W_0^{1,q}(Q)$ we have

$$|(\overline{A}u_1-\overline{A}u_2,v)|=|\int_Q(\overline{a}(x,Du_1)-\overline{a}(x,Du_2))\cdot Dv dx|\le \|\overline{a}(x,Du_1)-\overline{a}(x,Du_2)\|_{L^p(Q)}\|Dv\|_{L^q(Q)}\le k\|D(u_1-u_2)\|_{L^p(Q)}\|Dv\|_{L^q(Q)}. \tag{19}$$

This means that

$$\|\overline{A}u_1-\overline{A}u_2\|_{W^{-1,p}(Q)}\le k\|u_1-u_2\|_{W_0^{1,p}(Q)},$$

i.e., $\overline{A}:W_0^{1,p}(Q)\to W^{-1,p}(Q)$ is Lipschitz continuous with Lipschitz constant $k$, $k<1$. Q.E.D.

Now we define the operator $Q_h=Q_{h,f}$ ($f$ is fixed in $W^{-1,p}(Q)$ for some $p\ge 2$), $Q_h:E_h\to E_h$, by the formula ($v_h\in E_h$)

$$Q_hv_h=L_h(\overline{A}v_h+tf)=v_h-tL_h(Av_h-f).$$

The last equality follows from (15).

If $u_h\in E_h$ is a fixed point of $Q_h$, then $u_h$ is the approximate solution of $Au=f$ (easy to check). We consider $E_h$ with the norm induced from $W_0^{1,p}(Q)$.

**Lemma 2.** *$Q_h$ is Lipschitz continuous with the Lipschitz constant $M_pk$,*

$$\|Q_hu_h-Q_hv_h\|_{W_0^{1,p}(Q)}\le M_pk\|u_h-v_h\|_{W_0^{1,p}(Q)}.$$

**Proof.** Indeed,

$$\|Q_hu_h-Q_hv_h\|_{W_0^{1,p}(Q)}=\|L_h(\overline{A}u_h-\overline{A}v_h)\|_{W_0^{1,p}(Q)}\le M_p\|\overline{A}u_h-\overline{A}v_h\|_{W^{-1,p}(Q)}\le M_pk\|u_h-v_h\|_{W_0^{1,p}(Q)} \tag{20}$$

Note that $k<1$. Q.E.D.

Inequality (6) implies that if $s$ is sufficiently close to 2, then $M_p$ is close to 1 for all $p\in[2,s]$. Hence, $M_pk<1$ for $p\in[2,s]$ with $s$ close to 2.

Next we take $f,g\in W^{-1,p}(Q)$. Let $u_h$ and $w_h$ be approximate solution of

$$Au=f,\quad Aw=g.$$

Then $u_h$ and $w_h$ are fixed points of $Q_{h,f}$ and $Q_{h,g}$, respectively, and we have

$$\|u_h-w_h\|_{W_0^{1,p}(Q)}=\|Q_{h,f}u_h-Q_{h,g}w_h\|_{W_0^{1,p}(Q)}\le M_pk\|u_h-w_h\|_{W_0^{1,p}(Q)}+ \|Q_{h,f}w_h-Q_{h,g}w_h\|_{W_0^{1,p}(Q)}\le M_pk\|u_h-w_h\|_{W_0^{1,p}(Q)}+M_pt\|f-g\|_{W^{-1,p}(Q)}. \tag{21}$$

Hence,

$$\|u_h - w_h\|_{W_0^{1,p}(Q)} \le \frac{tM_p}{1 - M_p k}\|f - g\|_{W^{-1,p}(Q)}.$$

With $g = 0$ we have

$$\|u_h\|_{W_0^{1,p}(Q)} \le \frac{tM_p}{1 - M_p k}\|f\|_{W^{-1,p}(Q)}. \tag{22}$$

This completes the proof of the theorem.

5. **General nonlinear elliptic operators.** Consider

$$Au = -div(a(x, u, Du)) + a_0(x, u, Du)$$

and assume

- $a : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $a_0 : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are Carathéodory functions, and for simplicity we assume $a(x, 0, 0) = 0$, $a_0(x, 0, 0) = 0$.
- $$|a(x, \eta, \xi)| + |a_0(x, \eta, \xi)| \le C(1 + |\eta| + |\xi|).$$
- $$|a(x, \eta, \xi_1) - a(x, \eta, \xi_2)| \le M|\xi_1 - \xi_2|.$$
- $$(a(x, \eta, \xi_1) - a(x, \eta, \xi_2)) \cdot (\xi_1 - \xi_2) \ge m|\xi_1 - \xi_2|^2.$$
- $$a(x, \eta, \xi) \cdot \xi + a_0(x, \eta, \xi)\eta \ge \alpha|\xi|^2 - \beta,$$
  $\alpha > 0$, $\beta \ge 0$.

$A : W_0^{1,2}(Q) \to W^{-1,2}(Q)$ is a continuous pseudomonotone [11] (type $S_+$ [17]) coercive operator. Hence, $Au = f$, $f \in W^{-1,2}(Q)$ has a solution in $W_0^{1,2}(Q)$ (not necessarily unique). Consider an approximate problem

$$(Au_h, v_h) = (f, v_h), \ \forall v_h \in E_h.$$

The approximate problem has a solution $u_h \in E_h$ (not necessarily unique) and

$$\|u_h\|_{W_0^{1,2}(Q)} \le C, \ \forall \ h. \tag{23}$$

This follows from the properties of the operator $A$.

Introduce the operators $A_h$

$$A_h u = -diva(x, u_h, Du)$$

and

$$f_h = f - a_0(x, u_h, Du_h).$$

$A_h$ is a strongly monotone operator with operator constants independent of $h$ and the estimate (23) implies that

$$\|a_0(x, u_h, Du_h)\|_{L^2(Q)} \le C.$$

Because $L^2(Q) \subset W^{-1,p}(Q)$, $p \in [2, s]$, for some $s$, we have

$$\|f_h\|_{W^{-1,p}(Q)} \le C$$

uniformly, provided $f \in W^{-1,p}(Q)$.

Clearly, $u_h$ is an approximate solution of $A_h u_h = f_h$, which is understood in a variational sense with discrete test functions, $(A_h u_h, v_h) = (f_h, v_h)$, $\forall v_h \in E_h$. (14) implies that

$$\|u_h\|_{W^{1,p}(Q)} \le C, \ \forall \ h, \tag{24}$$

$p \in [2, s]$, $s$ is close to 2.

In the next section we will apply (24) to a numerical scheme. We would like to note that $u_h$ does not have to be the solution of standard Galerkin approximations of (25).

6. **An Application.** Consider the equation, $u \in W_0^{1,2}(Q)$

$$-div(a(x, u, Du)) + a_0(x, u, Du) = f. \tag{25}$$

Let $a$ and $a_0$ satisfy the assumptions imposed in the previous section and also the following assumption. For any $\xi, \xi' \in \mathbb{R}^n$, and $\eta, \eta' \in \mathbb{R}$

$$\begin{gathered} |a(x, \eta, \xi) - a(x, \eta', \xi')| + |a_0(x, \eta, \xi) - a_0(x, \eta', \xi')| \leq \\ C(1 + |\eta| + |\eta'| + |\xi| + |\xi'|)\nu(|\eta - \eta'|)+ \\ C(1 + |\eta|^{1-s} + |\eta'|^{1-s} + |\xi|^{1-s} + |\xi'|^{1-s})|\xi - \xi'|^s, \end{gathered} \tag{26}$$

for all $x \in Q$, where $0 < s < 1$, $\nu(r)$ is continuity modulus (i.e., a nondecreasing continuous function on $[0, +\infty)$ such that $\nu(0) = 0$, $\nu(r) > 0$ if $r > 0$, and $\nu(r) = 1$ if $r > 1$).

The equation (25) has a solution, and in this section we will be interested in the approximation of these solutions. Introduce

$$S^h = \{v_h \in C^0(\overline{Q}) : \text{the restriction } v_h \text{ is linear for each triangle } K \in \Pi_h \\ \text{and } v_h = 0 \text{ on } \partial Q\},$$

where $diam(K) \leq Ch$ and $\Pi_h$ is a standard triangulation of $Q$. We seek an approximation of a solution of (25), $u_h \in S^h$, such that

$$(A^h u_h, v_h) = (f, v_h), \ \forall v_h \in S^h, \tag{27}$$

where

$$(A^h u_h, v_h) = \int_Q a(x, M_h u_h, Du_h) \cdot Dv_h dx + \int_Q a_0(x, M_h u_h, Du_h) v_h dx.$$

Here $M_h$ is an averaging operator over each element $K \in \Pi_h$ defined as

$$M_h u_h = \sum_{K \in \Pi_h} 1_K \frac{1}{K} \int_K u_h dx, \tag{28}$$

where $1_K$ is an indicator function of $K$. Moreover, for any $\phi \in L^p(Q)$, $M_h\phi \to \phi$ in $L^p(Q)$. Note that the discretization (27) can be more tractable for computational purposes if the spatial dependence is not present because the quadrature step can be easily implemented.

Define $Au_h$ by

$$(Au_h, v_h) = \int_Q a(x, u_h, Du_h) \cdot Dv_h dx + \int_Q a_0(x, u_h, Du_h) v_h dx.$$

**Theorem 2.** *$u_h$ converges to $u$ in $W_0^{1,2}(Q)$ as $h \to 0$ along a subsequence, where $u_h$ is a solution of (27) and $u$ is a solution of (25).*

The proof of the theorem will be carried out in the following way. First, we will show the coercivity of the discrete operator, then the uniform boundedness of the solutions in $W_0^{1,2}(Q)$ and $W_0^{1,2+\alpha}(Q)$, for some $\alpha > 0$, will be shown. Further, the consistency of the discrete scheme will be investigated. Finally, to prove the theorem we will need the fact that the solutions are in $W_0^{1,2+\alpha}(Q)$.

The next lemma will be also used in the proof.

**Lemma 3.** *If $u_k \to 0$ in $L^r(Q)$ $(1 < r < \infty)$ as $k \to \infty$ then*

$$\int_Q \nu(u_k)|v_k|^p dx \to 0, \ \text{as } k \to \infty$$

*for all $v_k$ either (A) compact in $L^p(Q)$ or (B) bounded in $L^{p+\alpha}(Q)$, $\alpha > 0$. Here $\nu(r)$ is continuity modulus defined previously (see (26)) and $1 < p < \infty$.*

**Proof.** Because $u_k$ converges in $L^r$, it converges in measure. Consequently, for any $\delta > 0$ there exists $Q_\delta$ and $k_0$ such that $meas(Q \setminus Q_\delta) < \delta$ and $\nu(u_k) < \delta$ in $Q_\delta$ for $k > k_0$. Thus,

$$\int_Q \nu(u_k)|v_k|^p dx = \int_{Q_\delta} \nu(u_k)|v_k|^p dx + \int_{Q\setminus Q_\delta} \nu(u_k)|v_k|^p dx \leq C\delta + C\int_{Q\setminus Q_\delta} |v_k|^p dx. \tag{29}$$

Next, we use the fact that if (A) or (B) is satisfied then the set $v_k$ has equi-absolute continuous norm [10] (i.e., for any $\epsilon > 0$ there exists $\zeta > 0$ such that $meas(Q_\zeta) < \zeta$ implies $\|P_{Q_\zeta} v_k\|_p < \epsilon$ for all $k$ , where $P_D f = \{f(x), \text{ if } x \in D; 0 \text{ otherwise}\}$). Consequently, the second term on the right side of (29) converges to zero as $\delta \to 0$. Q.E.D.

To prove the theorem, we first show that $A_h$ is coercive.

**Lemma 4.** *$A^h$ is coercive for sufficiently small $h$, i.e.,*

$$(A^h u_h, u_h) \geq C\|u_h\|^2_{W_0^{1,2}(Q)} - C_0.$$

**Proof.**

$$\begin{aligned}
&(A^h u_h, u_h) = \int_Q a(x, M_h u_h, Du_h)\cdot Du_h dx + \int_Q a_0(x, M_h u_h, Du_h) u_h dx = \\
&\int_Q a(x, M_h u_h, Du_h)\cdot Du_h dx + \int_Q a_0(x, M_h u_h, Du_h) M_h u_h dx + \\
&\int_Q a_0(x, M_h u_h, Du_h)(u_h - M_h u_h) dx \geq C\int_Q |Du_h|^2 dx - C_0 - \\
&|\int_Q a_0(x, M_h u_h, Du_h)(u_h - M_h u_h) dx| \geq C\int_Q |Du_h|^2 dx - \\
&C_2 h \int_Q |Du_h|^2 dx - C_0 = (C - C_2 h)\int_Q |Du_h|^2 dx - C_0.
\end{aligned} \tag{30}$$

Here, we have used the fact that $|u_h - M_h u_h| < Ch|Du_h|$ in every triangular element $K$.

Q.E.D.

It can be easily shown that $A^h$ is continuous, which guarantees the existence of the discrete solutions [15]. Indeed, one can easily show that $Fu_h = A^h u_h - f$ satisfies $(Fu_h, u_h) \geq 0$ for $\|u_h\| > \rho$, for some $\rho > 0$, consequently $F$ has a zero [15]. Moreover, because of the coerciveness we have the following uniform bound

$$\|u_h\|_{W_0^{1,2}(Q)} \leq C,$$

where $u_h$ are solutions of (27). As a consequence, $u_h \to u$ weakly in $W_0^{1,2}(Q)$ (along a subsequence) as $h \to 0$. For further analysis, the sequence $u_h$ is fixed. The next lemma is important for the proof of the theorem.

**Lemma 5.**

$$(A^h u_h - Au_h, v_h) \to 0,$$

*for any uniformly bounded family of* $u_h$ *and compact family of* $v_h$ *in* $W_0^{1,2}(Q)$. *Moreover, if* $u_h$ *is uniformly bounded in* $W_0^{1,2+\alpha}(Q)$ $(\alpha > 0)$ *then*

$$(A^h u_h - Au_h, u_h) \to 0. \tag{31}$$

**Proof.** Consider

$$\begin{aligned}(A^h u_h - Au_h, v_h) = \int_Q ((a(x, M_h u_h, Du_h) - a(x, u_h, Du_h)) \cdot Dv_h + \\ (a_0(x, M_h u_h, Du_h) - a_0(x, u_h, Du_h))v_h)dx.\end{aligned} \tag{32}$$

Using the estimate (26), we have

$$\begin{aligned}&|\int_Q (a(x, M_h u_h, Du_h) - a(x, u_h, Du_h)) \cdot Dv_h dx| \leq \\ &C \int_Q (1 + |M_h u_h| + |Du_h| + |u_h|)\nu(|M_h u_h - u_h|)|Dv_h|dx \leq \\ &C(\int_Q (1 + |u_h|^2 + |Du_h|^2)dxdt)^{1/2}(\int_Q |Dv_h|^2 \nu(|M_h u_h - u_h|)^2 dx)^{1/2} \leq \\ &C(1 + \|u_h\|^2_{W_0^{1,2}(Q)})^{1/2}(\int_Q |Dv_h|^2 \nu(h|Du_h|)^2 dx)^{1/2}.\end{aligned} \tag{33}$$

Here we have used $|u_h - M_h u_h| \leq Ch|Du_h|$. Because of Lemma 3, we obtain that the right side of (33) converges to zero for any uniformly bounded family of $u_h \in W_0^{1,2}(Q)$ and compact family $v_h \in W_0^{1,2}(Q)$ as $h \to 0$. The estimate for $a_0$ can be obtained in a similar way,

$$\begin{aligned}&|\int_Q (a_0(x, M_h u_h, Du_h) - a_0(x, u_h, Du_h))v_h dx| \leq \\ &(C + \|u_h\|^2_{W_0^{1,2}(Q)})^{1/2}(\int_Q |v_h|^2 \nu(h|Du_h|)^2 dx)^{1/2}.\end{aligned} \tag{34}$$

Note that the right side of (34) converges to zero for any uniformly bounded family of $u_h \in W_0^{1,2}(Q)$ and $v_h \in W_0^{1,2}(Q)$. Indeed, the latter implies that $v_h$ is uniformly bounded in $L^{2+\alpha}(Q)$ for some $\alpha > 0$. Thus applying Lemma 3, we obtain that the right side of (34) converges to zero for any uniformly bounded family of $v_h$ in $W_0^{1,2}(Q)$.

To show (31) we note that

$$(A^h u_h - Au_h, u_h) \leq C(1 + \|u_h\|^2_{W_0^{1,2}(Q)})^{1/2}(\int_Q |Du_h|^2 \nu(h|Du_h|)^2 dx)^{1/2}. \tag{35}$$

Because $Du_h$ is uniformly bounded in $L^{2+\alpha}(Q)$, $\alpha > 0$ we obtain that the right side of (35) converges to zero according to lemma 3. Q.E.D.

**Lemma 6.** *For some* $\alpha > 0$ *we have*

$$\|Du_h\|_{W_0^{1,2+\alpha}(Q)} \leq C.$$

**Proof.** To prove this lemma we use the results of the previous section. Consider the operator,

$$A_0^h u = -div(a(x, M_h u_h, Du))$$

and

$$f_h = f - a_0(x, M_h u_h, Du_h),$$

where $u_h$ is a discrete solution of (27). Then $A_0^h$ is strongly monotone with operator constants independent of $h$. Moreover, using the previous lemma we have that

$$\|a_0(x, M_h u_h, Du_h)\|_{L^2(Q)} \le C.$$

Clearly $u_h$ is a solution of $A_0^h u_h = f_h$ (understood again in a variational sense with discrete test functions), and thus we have

$$\|u_h\|_{W_0^{1,2+\alpha}(Q)} \le C,$$

for some $\alpha > 0$. Q.E.D.

**Lemma 7.** *$Au_h \to f$ weakly in $W^{-1,q}(Q)$ as $h \to 0$, where $u_h$ are solutions of (27).*

**Proof.** Indeed, for any $v \in W_0^{1,2}(Q)$ and $v_h \to v$ in $W_0^{1,2}(Q)$ we have

$$\lim_{h\to 0}(Au_h, v_h) = \lim_{h\to 0}(A^h u_h, v_h) + \lim_{h\to 0}(Au_h - A^h u_h, v_h) = \lim_{h\to 0}(f, v_h) = (f, v).$$

Here we have used Lemma 5. Q.E.D.

Thus, we have the following:

$$u_h \to u \text{ weakly in } W_0^{1,p}(Q), \quad Au_h \to f \text{ weakly in } W^{-1,p}(Q), \quad (Au_h, u_h) \to (f, u).$$

Because the operator $A$ is type $M$, [15] this guarantees that $Au = f$, i.e., $u$ is a solution. Moreover, because our differential operators are also type $S_+$ [17], we have $u_h \to u$ strongly $W_0^{1,2}(Q)$. This completes the proof of the theorem.

6.1. **Generalization.** One can generalize the above numerical procedure for (25). In particular, let $S^h$ and $E^h$ be families of finite dimensional subspaces such that $span(\bigcup S^h)$ and $span(\bigcup E^h)$ are dense in $W_0^{1,2}(Q)$, i.e., for any $v \in W_0^{1,2}(Q)$ there exists a family of $v_h$ ($v_h \in S^h$ or $v_h \in E^h$) such that $v_h \to v$ in $W_0^{1,2}(Q)$. Consider operators $M_h$ acting from $S^h$ to $E^h$ such that

$$\|M_h u_h - u_h\|_{L^2(Q)} \to 0 \tag{36}$$

as $h \to 0$ for any $u_h \in S^h$, such that $\|u_h\|_{W_0^{1,2}(Q)} \le C$. Note that $M_h u_h$ are not necessarily in $S^h$ and if $M_h$ is defined by (28) then (36) holds. If $M_h$ is defined in the whole space of $W_0^{1,2}(Q)$ then $M_h$ is an approximation of the identity defined on $W_0^{1,2}(Q)$. Consider the following discretization of (25). Seek $u_h \in S^h$ such that

$$\int_Q (a(x, M_h u_h, Du_h) \cdot Dv_h + a_0(x, M_h u_h, Du_h) v_h) dx = \int_Q f v_h dx.$$

Our analysis can be modified to obtain $u_h \to u$ in $W_0^{1,2}(Q)$ (cf. (30) and (33)).

7. **Acknowledgments.** We would like to thank the referees for their valuable comments and suggestions. The research of the first author is partially supported by NSF.

## REFERENCES


[1] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[2] S. C. Brenner and L. R. Scott, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer-Verlag, New York, first ed., 1994.

[3] Y. Efendiev and A. Pankov, *Numerical homogenization and correctros for random nonlinear elliptic operators*, SIAM J. Applied Math., 65(1) (2004), pp. 43–68.

[4] ———, *Numerical homogenization of nonlinear random parabolic operators*, SIAM Multiscale Modeling and Simulation, 2(2) (2004), pp. 237–268.

[5] K. Gröger, *A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations*, Math. Ann., 283 (1989), pp. 679–687.

[6] K. Gröger and A. Pankov, *Meyers type regularity for bounded and almost periodic solutions to nonlinear second order parabolic equations with mixed boundary conditions*, Mat. Fiz. Anal. Geom., 3 (1996), pp. 46–64.

[7] R. Helmig, *Multiphase Flow and Transport Processes in the Subsurface*, Springer, New York, 1997.

[8] A. I. Koševev, *Regularity of the solutions of quasilinear elliptic systems*, Uspekhi Mat. Nauk, 33 (1978), pp. 3–49, 255.

[9] A. I. Košelev, *Regularity problem for quasilinear elliptic and parabolic systems*, "Springer", Berlin, New York, 1995.

[10] M. A. Krasnosel′skiĭ, P. P. Zabreĭko, E. I. Pustyl′nik, and P. E. Sobolevskiĭ, *Integral operators in spaces of summable functions*, Noordhoff International Publishing, Leiden, 1976. Translated from the Russian by T. Ando, Monographs and Textbooks on Mechanics of Solids and Fluids, Mechanics: Analysis.

[11] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.

[12] N. G. Meyers and A. Elcrat, *Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions*, Duke Math. J., 42 (1975), pp. 121–136.

[13] A. Pankov, *G-convergence and homogenization of nonlinear partial differential operators*, Kluwer Academic Publishers, Dordrecht, 1997.

[14] L. Richards, *Capillary conduction of liquids through porous mediums*, Physics, (1931), pp. 318–333.

[15] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, vol. 49 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1997.

[16] C. G. Simader, *On Dirichlet's boundary value problem*, Springer-Verlag, Berlin, 1972. An $L^p$-theory based on a generalization of Gårding's inequality, Lecture Notes in Mathematics, Vol. 268.

[17] I. V. Skrypnik, *Methods for analysis of nonlinear elliptic boundary value problems*, vol. 139 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1994. Translated from the 1990 Russian original by Dan D. Pascali.


*E-mail address*: efendiev@math.tamu.edu; pankov@math.wm.edu