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Meyers type estimates for approximate solutions of nonlinear parabolic equations and their applications

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Abstract — In this paper we obtain Meyers type ($L^{p+\varepsilon}$-) estimates for approximate solutions of nonlinear parabolic equations. This research is motivated by a numerical homogenization of these type of equations [**?**]. Using derived estimates we show the convergence of numerical solutions obtained from numerical homogenization methods.

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1. INTRODUCTION

2. INTRODUCTION

Meyers type regularity estimates [**?**] provide extra integrability of the solutions of differential equations. This property of the solutions is found to be useful in many applications. In this paper our goal is to derive Meyers type estimates for the approximate (numerical) solutions. The need for such estimates arises, for example, in the numerical homogenization of nonlinear parabolic equations (see [**?**], page 255). Only extra integrability of the discrete solutions guarantees their convergence to the solution of continuous equations under suitable assumptions.

In this paper we are interested in obtaining Meyers type ($L^{p+\varepsilon}$-) estimates for numerical solutions of nonlinear parabolic equations. Our basic tool is the technique presented in [**?**,**?**] for continuous operators. The starting point of our approach is the use of Meyers type estimates for linear heat equations. Further, using adequate contraction maps we extend these estimates to strongly monotone operators. In the next step, using a particular discrete solution we obtain Meyers type estimates for more general parabolic operators of the form

$$D_t u - div(\mathbf{a}(x,u,D_x u)) + a_0(x,u,D_x u) = f \tag{2.1}$$

with homogeneous Dirichlet boundary conditions under quite general assumptions

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on **a** and a_0. To obtain Meyers type estimates for (2.1) we use weaker assumptions than those imposed in [**?**].

In the paper we also consider an application of the obtained estimates to a particular discretization of (2.1) which arises in a numerical homogenization of such equations [**?**]. This discretization of (2.1) is different from the standard Galerkin discretization. In particular, two different discrete spaces are involved for approximation of u and $D_x u$. This scenario can not be avoided in a numerical homogenization procedure because the solution of the local problems do not belong to the discrete spaces that are used for approximation of the homogenized solutions [**?**]. To obtain the convergence of the discrete solutions to a solution of (2.1) one needs Meyers type estimates.

The paper is organized as follows. In the next section we discuss preliminary results regarding linear equations. In the following section we obtain the estimates for monotone operators. Section 4 is devoted to the Meyers type estimates for equations (2.1). Finally we use these estimates to prove the convergence of a numerical scheme.

3. PRELIMINARIES

Denote $Q = [0,T] \times Q_0$ and

$$\begin{aligned} X^p = L^p(0,T,W_0^{1,p}(Q_0)), \quad X^{-p} = L^p(0,T,W^{-1,p}(Q_0)) \\ Y^p = L^p(0,T,L^p(Q_0)), \quad W^p = \{u \in X^p,\ D_t u \in X^{-p},\ u(t=0)=0\}, \end{aligned} \tag{3.1}$$

where $\|u\|_{W^p} = \|u\|_{X^p} + \|D_t u\|_{X^{-p}}$. Throughout the paper C (with or without sub-(or super-)script) denotes a generic constant which independent of h, unless otherwise stated.

Let E_h be a family of finite dimensional subspaces such that $span(\bigcup E_h)$ is dense in $W_0^{1,p}(Q_0)$, $2 \leqslant p \leqslant p_0$ (some fixed p_0). Consider $u_h(t) \in E_h$, $u_h(t=0)=0$,

$$\lambda(D_t u_h, v_h) + (D_x u_h, D_x v_h) = (f, v_h),\ \forall v_h \in E_h, \tag{3.2}$$

where λ is a parameter. (u,v) means $\int_{Q_0} uvdx$ and the duality pairing between $W^{1,p}(Q_0)$ and $W^{-1,q}(Q_0)$, and $1/p + 1/q = 1$. Our *basic assumption* is that for any $f \in X^{-p}$, for some p, $2 \leqslant p \leqslant p_0$, we have

$$\|u_h\|_{X^p} \leqslant C\|f\|_{X^{-p}}. \tag{3.3}$$

$C = C_{p,\lambda}$ depends on p and λ, but not on h. One can assume (3.3) only for $p \in [2,s]$, for some $s > 2$. This result for $p = 2$ can be easily obtained and known (e.g., [**?**,**?**]). Moreover, $C_{2,\lambda} \leqslant 1$. In the paper we assume Meyers type estimates for linear heat equation. (3.3) and (3.2) implies that $\|u_h\|_{W^p} \leqslant C_{p,\lambda}\|f\|_{X^{-p}}$, but in general $C_{p,\lambda} > 1$.

Denote by L_h^λ the linear operator that maps f into u_h,

$$L_h^\lambda f = u_h.$$

Inequality (3.3) implies that L_h^λ is a bounded linear operator from X^{-p} into W^p, as well as from X^{-p} to X^p. Denote by $M_{p,\lambda,h}$ the norm of the operators L_h^λ from X^{-p} to X^p and set

$$M_{p,\lambda} = \sup_h M_{p,\lambda,h}.$$

It can be easily shown that $M_{2,\lambda} \leqslant 1$.

Theorem 3.1. *Let $p_0 > s > 2$, $2 \leqslant p \leqslant s$ and*

$$\frac{1}{p} = \frac{1-\vartheta}{2} + \frac{\vartheta}{s}.$$

Then

$$M_{p,\lambda} \leqslant (M_{s,\lambda})^\vartheta. \tag{3.4}$$

Proof.
Introduce a family of operators B_h^λ,

$$B_h^\lambda = D \circ L_h^\lambda \circ div.$$

The operator B_h^λ is a bounded linear operator that acts in the space Y^p. Next we estimate the norm of B_h^λ in Y^s.

$$\|B_h^\lambda\|_{Y^s} = \sup_{\|u\|_{L^s(Q)} \leqslant 1} \|D \circ L_h^\lambda \circ div(u)\|_{Y^s} = \sup_{\|f\|_{X^{-s}} \leqslant 1} \|D \circ L_h^\lambda \circ f\|_{Y^s} = \sup_{\|f\|_{X^{-s}} \leqslant 1} \|L_h^\lambda \circ f\|_{X^s} = \|L_h^\lambda\|_{X^s} \leqslant M_{s,\lambda,h}. \tag{3.5}$$

Similarly we can check that $\|B_h^\lambda\|_{Y^2} \leqslant M_{2,\lambda,h} \leqslant 1$. Next we apply the Riesz-Thorin interpolation theorem [**?**] on B_h^λ. Let $s > 2$, $2 \leqslant p \leqslant s$ and

$$\frac{1}{p} = \frac{1-\vartheta}{2} + \frac{\vartheta}{s}.$$

Then using Riesz-Thorin interpolation theorem we have

$$\|B_h^\lambda\|_{Y^p} \leqslant (M_{s,h})^\vartheta. \tag{3.6}$$

Introduce z

$$(z, D_x v) = (f, v),\ \forall v \in X^2.$$

Since $(z, D_x v) \leqslant \|f\|_{X^{-p}} \|v\|_{X^p}$, z is a continuous linear functional on a subspace of Y^q. By the Hahn-Banach theorem, z can be extended to a continuous linear functional on Y^q and $\|z\|_{Y^q} \leqslant \|f\|_{X^{-p}}$. Moreover, $div(z) = f$.

Denote $u_h = L_{h,\lambda} f$. Then $D_x u_h = B_h^\lambda z$ and

$$\|D_x u_h\|_{Y^p} \leqslant M_{s,\lambda}^{\vartheta} \|z\|_{Y^p} \leqslant M_{s,\lambda}^{\vartheta} \|f\|_{X^{-p}}.$$

Thus, $u_h \in X^p$ and

$$\|u_h\|_{X^p} \leqslant M_{s,\lambda}^{\vartheta} \|f\|_{X^{-p}}.$$

Q.E.D.

4. MONOTONE OPERATORS

Consider

$$D_t u + Au = f,$$

where

$$Au = -div(\mathbf{a}(x,t,D_x u))$$

and assume that

- $\mathbf{a} : Q \times R^n \to R^n$ be a Caratheodory function and $\mathbf{a}(x,t,0) = 0$ (the latter is for the sake of simplicity).
- $$|\mathbf{a}(x,t,\xi_1) - \mathbf{a}(x,t,\xi_2)| \leqslant M|\xi_1 - \xi_2| \tag{4.1}$$
  for all $(x,t) \in Q$ and $\xi_1, \xi_2 \in R^n$.
- $$(\mathbf{a}(x,t,\xi_1) - \mathbf{a}(x,t,\xi_2)) \cdot (\xi_1 - \xi_2) \geqslant m|\xi_1 - \xi_2|^2. \tag{4.2}$$

A is a strictly monotone continuous operator from X^2 to X^{-2}, and

$$(Au - Av, u - v) \geqslant m\|u - v\|_{X^2}^2.$$

Hence, it is coercive. In fact A maps X^2 continuously into X^{-2},

$$\|Au - Av\|_{X^{-2}} \leqslant M\|u - v\|_{X^2}.$$

Consider an approximate problem. Find $u_h \in E_h$ such that

$$(D_t u_h, v_h) + (Au_h, v_h) = (f, v_h),\ \forall v_h \in E_h. \tag{4.3}$$

For the analysis we will need the following notation

$$\lambda = \frac{m}{M^2}, \quad k = (1 - \frac{m^2}{M^2})^{1/2}.$$

Our goal is to prove the following theorem.

Theorem 4.1. *Let u_h be the solution of (4.3). Then*

$$\|u_h\|_{W^p} \leqslant C\|f\|_{X^{-p}}. \tag{4.4}$$

For the proof of the theorem we will need some auxiliary facts. Consider the operator $\overline{A}$ defined by

$$\overline{A}u = -\Delta u - \lambda A(u).$$

Lemma 4.1.

$$\|\overline{A}u_1 - \overline{A}u_2\|_{X^{-p}} \leqslant k\|u_1 - u_2\|_{X^p},$$

i.e., $\overline{A} : X^p \to X^{-p}$ is Lipschitz continuous with Lipschitz constant k, $k < 1$.

Proof.

The flux corresponding to $\overline{A}$ is given by

$$\overline{\mathbf{a}}(x,t,\xi) = \xi - \lambda\mathbf{a}(x,t,\xi).$$

Next we would like to derive the following estimate for $\overline{\mathbf{a}}(x,\xi)$,

$$|\overline{\mathbf{a}}(x,t,\xi_1) - \overline{\mathbf{a}}(x,t,\xi_2)| \leqslant (1 - \frac{m^2}{M^2})^{1/2}|\xi_1 - \xi_2| = k|\xi_1 - \xi_2|. \tag{4.5}$$

Indeed,

$$\begin{aligned}|\overline{\mathbf{a}}(x,t,\xi_1) - \overline{\mathbf{a}}(x,t,\xi_2)|^2 = (\xi_1 - \xi_2)\cdot(\xi_1 - \xi_2) - 2\lambda(\mathbf{a}(x,t,\xi_1) - \mathbf{a}(x,t,\xi_2))\cdot(\xi_1 - \xi_2) + \\ \lambda^2(\mathbf{a}(x,t,\xi_1) - \mathbf{a}(x,t,\xi_2))\cdot(\mathbf{a}(x,t,\xi_1) - \mathbf{a}(x,t,\xi_2)) = \\ |\xi_1 - \xi_2|^2 - 2\lambda(\mathbf{a}(x,t,\xi_1) - \mathbf{a}(x,t,\xi_2))\cdot(\xi_1 - \xi_2) + \lambda^2|\mathbf{a}(x,t,\xi_1) - \mathbf{a}(x,t,\xi_2)|^2.\end{aligned} \tag{4.6}$$

Assumptions (4.1) and (4.2) imply

$$\begin{aligned}|\overline{\mathbf{a}}(x,t,\xi_1) - \overline{\mathbf{a}}(x,t,\xi_2)|^2 \leqslant |\xi_1 - \xi_2|^2 - 2\lambda m|\xi_1 - \xi_2|^2 + \lambda^2 M^2|\xi_1 - \xi_2|^2 = \\ (1 - 2\lambda m + \lambda^2 M^2)|\xi_1 - \xi_2|^2 = k^2|\xi_1 - \xi_2|^2.\end{aligned} \tag{4.7}$$

The estimate (4.5) implies immediately that for any $u_1, u_2 \in W^p$ and $v \in W^q$ we have

$$\begin{aligned}|(\overline{A}u_1 - \overline{A}u_2, v)| = |\int_Q (\overline{\mathbf{a}}(x,t,D_x u_1) - \overline{\mathbf{a}}(x,t,D_x u_2))\cdot D_x v dx| \leqslant \\ \|\overline{\mathbf{a}}(x,t,D_x u_1) - \overline{\mathbf{a}}(x,t,D_x u_2)\|_{L^p(Q)}\|D_x v\|_{L^q(Q)} \leqslant k\|D(u_1 - u_2)\|_{Y^p}\|D_x v\|_{Y^q}.\end{aligned} \tag{4.8}$$

This means that

$$\|\overline{A}u_1 - \overline{A}u_2\|_{X^{-p}} \leqslant k\|u_1 - u_2\|_{X^p},$$

i.e., $\overline{A} : W^p \to X^{-p}$ is Lipschitz continuous with Lipschitz constant k, $k < 1$.

Q.E.D.

Now we define the operator $Q_h = Q_{h,f}$ (f is fixed in $W^{-1,p}(Q)$ for some $p \geqslant 2$), $Q_h : E_h \to E_h$, by the formula ($v_h \in E_h$)

$$Q_h v_h = L_h^\lambda(\overline{A}v_h + \lambda f)$$

If $u_h \in E_h$ is a fixed point of Q_h, then u_h is the approximate solution of $D_t u + Au = f$. Indeed, from $L_h^\lambda(\overline{A}v_h + \lambda f) = v_h$ it follows that $(\overline{A}v_h + \lambda f, w_h) = (\lambda D_t v_h - \Delta v_h, w_h)$ $(\forall w_h \in E_h)$, which is equivalent to $(-\lambda A v_h + \lambda f, w_h) = (\lambda D_t v_h, w_h)$. The latter implies that v_h is the solution of $(D_t v_h, w_h) + (A v_h, w_h) = (f, w_h)$. We consider E_h with the norm induced from W^p.

Lemma 4.2. *Q_h is Lipschitz continuous with the Lipschitz constant $M_p k$ from X^p to itself,*

$$\|Q_h u_h - Q_h v_h\|_{X^p} \leqslant M_p k \|u_h - v_h\|_{X^p}.$$

Proof. Indeed,

$$\|Q_h u_h - Q_h v_h\|_{X^p} = \|L_h^\lambda(\overline{A}u_h - \overline{A}v_h)\|_{X^p} \leqslant M_p\|\overline{A}u_h - \overline{A}v_h\|_{X^{-p}} \leqslant M_p k\|u_h - v_h\|_{X^p} \tag{4.9}$$

Note that $k < 1$.

Q.E.D.

Proof. (Theorem 4.1).

Inequality (3.4) implies that if s is sufficiently close to 2, then M_p is close to 1 for all $p \in [2, s]$. Hence $M_p k < 1$ for $p \in [2, s]$ with s close to 2.

Next we take $f, g \in X^{-p}$. Let u_h and w_h be the approximate solutions of

$$Au = f, \quad Aw = g.$$

Then u_h and w_h are fixed points of $Q_{h,f}$ and $Q_{h,g}$, respectively and we have

$$\|u_h - w_h\|_{X^p} = \|Q_{h,f}u_h - Q_{h,g}w_h\|_{X^p} \leqslant M_p k\|u_h - w_h\|_{X^p} + \|Q_{h,f}w_h - Q_{h,g}w_h\|_{X^p} \leqslant M_p k\|u - w\|_{X^p} + M_p\lambda\|f - g\|_{X^{-p}}. \tag{4.10}$$

Hence,

$$\|u_h - w_h\|_{X^p} \leqslant \frac{\lambda M_p}{1 - M_p k}\|f - g\|_{X^{-p}}.$$

Further using the fact that $(D_t u_h + Au_h, v_h) = (f, v_h)$ and $(D_t w_h + Aw_h, v_h) = (g, v_h)$ we have

$$\|D_t u_h - D_t w_h\|_{X^{-p}} = \|f - g - Au_h + Aw_h\|_{X^{-p}} \leqslant \|f - g\|_{X^{-p}} + M\|u_h - w_h\|_{X^p} \leqslant (1 + \frac{\lambda M M_p}{1 - M_p k})\|f - g\|_{X^{-p}}. \tag{4.11}$$

From these estimates we obtain

$$\|u_h - w_h\|_{W^p} \leqslant C\|f - g\|_{X^{-p}}.$$

With $g = 0$ we have

$$\|u_h\|_{W^p} \leqslant C\|f\|_{X^{-p}}. \tag{4.12}$$

This completes the proof of the theorem.
Q.E.D.

5. GENERAL NONLINEAR PARABOLIC OPERATORS

Consider

$$D_t u + Au = f,$$

where

$$Au = -div(\mathbf{a}(x,t,u,D_x u)) + a_0(x,t,u,D_x u)$$

and assume

- $\mathbf{a} : Q \times R \times R^n \to R^n$, $a_0 : Q \times R \times R^n \to R^n$ are Caratheodory functions, and for simplicity we assume $a(x,t,0,0) = 0$, $a_0(x,t,0,0) = 0$.
- $$|\mathbf{a}(x,t,\eta,\xi)| + |a_0(x,t,\eta,\xi)| \leqslant C(1 + |\eta| + |\xi|). \tag{5.1}$$
- $$|\mathbf{a}(x,t,\eta,\xi_1) - \mathbf{a}(x,t,\eta,\xi_2)| \leqslant M|\xi_1 - \xi_2|. \tag{5.2}$$
- $$(\mathbf{a}(x,t,\eta,\xi_1) - \mathbf{a}(x,t,\eta,\xi_2)) \cdot (\xi_1 - \xi_2) \geqslant m|\xi_1 - \xi_2|^2. \tag{5.3}$$
- $$\mathbf{a}(x,t,\eta,\xi) \cdot \xi + a_0(x,t,\eta,\xi)\eta \geqslant \alpha|\xi|^2 - \beta, \tag{5.4}$$
  $\alpha > 0$, $\beta \geqslant 0$.

$A : X^2 \to X^{-2}$ is a continuous pseudomonotone [**?**] (and type S_+ [**?**]) coercive operator. Hence $D_t u + Au = f,\ f \in X^{-2}(Q)$ has a solution in W^2 (not necessarily unique). Consider an approximate problem

$$(D_t u_h, v_h) + (Au_h, v_h) = (f, v_h),\ \forall v_h \in E_h. \tag{5.5}$$

The approximate problem has a solution $u_h \in E_h$ (not necessarily unique) and

$$\|u_h\|_{W^2} \leqslant C,\ \forall\, h. \tag{5.6}$$

This follows from the coerciveness of the operator A.

Theorem 5.1. *Let u_h be a solution of (5.5). Then*

$$\|u_h\|_{W^p} \leqslant C,\ \forall\ h,$$

$p \in [2,s]$, s is close to 2.

Proof.
Introduce the operators A_h

$$A_h u = -div\mathbf{a}(x, u_h, D_x u)$$

and

$$f_h = f - a_0(x, u_h, D_x u_h).$$

A_h is strongly monotone operator with operator constants independent of h and the estimate (5.6) implies that

$$\|a_0(x, u_h, D_x u_h)\|_{Y^2} \leqslant C.$$

Since $Y^2 \subset X^{-p}$, $p \in [2,s]$, for some s, we have

$$\|f_h\|_{X^{-p}} \leqslant C$$

uniformly, provided $f \in X^{-p}$.

Clearly, u_h is an approximate solution of $A_h u_h = f_h$ (i.e., $(A_h u_h, w_h) = (f, w_h)$, $\forall w_h \in E_h$) and (4.4) imply that

$$\|u_h\|_{W^p} \leqslant C,\ \forall\ h, \tag{5.7}$$

$p \in [2,s]$, s is close to 2.

Q.E.D.

In the next section we will apply (5.7) to a numerical scheme.

6. AN APPLICATION

Consider the equation, $u(t) \in W^2$

$$D_t u - div(\mathbf{a}(x,t,u,D_x u)) + a_0(x,t,u,D_x u) = f. \tag{6.1}$$

Let $\mathbf{a}$ and a_0 satisfy the assumptions (5.1)-(5.4) and also the following assumption, for any $\xi, \xi' \in R^n$ and $\eta, \eta' \in R$

$$|\mathbf{a}(x,t,\eta,\xi) - \mathbf{a}(x,t,\eta',\xi')| + |a_0(x,t,\eta,\xi) - a_0(x,t,\eta',\xi')| \leqslant$$
$$C(1+|\eta|+|\eta'|+|\xi|+|\xi'|)\nu(|\xi-\xi'|) + C(1+|\eta|^{1-s}+|\eta'|^{1-s}+|\xi|^{1-s}+|\xi'|^{1-s})|\xi-\xi'|^s, \tag{6.2}$$

for all $(x,t) \in R^{n+1}$, where $0 < s < 1$, $\nu(r)$ is continuity modulus (i.e., a nondecreasing continuous function on $[0,+\infty)$ such that $\nu(0)=0$, $\nu(r)>0$ if $r>0$, and $\nu(r)=1$ if $r>1$). Note that (6.1) with the imposed conditions (5.1)-(5.4) and (6.2) may have multiple solutions.

The equation (6.1) has a solution and in this section we will be interested in the approximation of this solution with the following discretization. Let E_h be given by

$$E_h = \{v_h \in C^0(\overline{Q}) : \text{the restriction } v_h \text{ is linear for each triangle } K \in \Pi_h \text{ and } v_h = 0 \text{ on } \partial Q\},$$

where Π_h is a standard triangular discretization of Q_0, $diam(K) \leqslant Ch$, We seek a solution, $u_h(t) \in E_h$, of (6.1) in E_h such that

$$(D_t u_h, v_h) + (A_h u_h, v_h) = (f, v_h), \ \forall v_h \in E_h, \tag{6.3}$$

where

$$(A_h u_h, v_h) = \int_{Q_0} \mathbf{a}(x,t,M_h u_h, D_x u_h) \cdot D_x v_h dx + \int_{Q_0} a_0(x,t,M_h u_h, D_x u_h) v_h dx. \tag{6.4}$$

Here M_h is an averaging operator over the each element $K \in \Pi_h$ defined as follows. For each $u_h \in E_h$,

$$M_h u_h = \sum_K \mathbf{1}_K \frac{1}{K} \int_K u_h dx, \tag{6.5}$$

where $\mathbf{1}_K$ is an indicator function of K. Moreover, for any $\varphi \in L^p(Q)$, $M_h \varphi \to \varphi$ in $L^p(Q)$ (see e.g., [**?**]). Note that the discretization (6.3) can be more tractable for computational purposes since the spatial dependence is not present and the quadrature step can be easily implemented.

Define Au_h on E_h by

$$(Au_h, v_h) = \int_{Q_0} \mathbf{a}(x,t,u_h, D_x u_h) \cdot D_x v_h dx + \int_{Q_0} a_0(x,t,u_h, D_x u_h) v_h dx. \tag{6.6}$$

Theorem 6.1. *u_h converges to u in X^2 as $h \to 0$ along a subsequence, where u_h is a solution of* (6.3) *and u is a solution of* (6.1)*.*

The proof of the theorem will be carried out in the following way. First we will show the coercivity of the discrete operator, then the uniform boundedness of the solutions in X^2 and $A_h u_h$ in X^{-2}. This will imply that $u_h \to u$ and $A_h u_h \to g$ weakly in the corresponding spaces. Then using standard technique for parabolic equations we obtain that $D_t u + g = f$ in X^{-2}. It remains to show that $g = Au$. The analysis of the latter requires Meyers type estimates.

Next lemma will be used in the proof of Theorem 6.1.

Lemma 6.1. *If $u_k \to 0$ in $L^r(Q)$ $(1 < r < \infty)$ as $k \to \infty$ then*

$$\int_Q \nu(u_k) |v_k|^p dxdt \to 0, \text{ as } k \to \infty$$

for all v_k either (1) compact in $L^p(Q)$ or (2) bounded in $L^{p+\alpha}(Q)$, $\alpha > 0$. Here $\nu(r)$ is continuity modulus defined previously (see (6.2)) and $1 < p < \infty$.

Proof. Since u_k converges in L^r it converges in measure. Consequently, for any $\delta > 0$ there exists Q_δ and k_0 such that $meas(Q \setminus Q_\delta) < \delta$ and $\nu(u_k) < \delta$ in Q_δ for $k > k_0$. Thus

$$\int_Q \nu(u_k)|v_k|^p dxdt = \int_{Q_\delta} \nu(u_k)|v_k|^p dxdt + \int_{Q \setminus Q_\delta} \nu(u_k)|v_k|^p dxdt \leqslant C\delta + C\int_{Q \setminus Q_\delta} |v_k|^p dxdt. \tag{6.7}$$

Next we use the fact that if (1) or (2) satisfies then the set v_k has equi-absolute continuous norm [**?**] (i.e., for any $\varepsilon > 0$ there exists $\zeta > 0$ such that $meas(Q_\zeta) < \zeta$ implies $\|P_{Q_\zeta} v_k\|_p < \varepsilon$, where $P_D f = \{f(x),\ \text{if } x \in D; 0 \text{ otherwise}\}$). Consequently, the second term on the right hand side of (6.7) converges to zero as $\delta \to 0$. Q.E.D.

To prove Theorem 6.1 we first show that A_h is coercive,

Lemma 6.2. *A_h is coercive for sufficiently small h, i.e.,*

$$(A_h u_h, u_h) \geqslant C\|u_h\|_{X^2} - C_0,$$

where C and C_0 are generic constants independent of h.

Proof.

$$\begin{aligned}
&(A_h u_h, u_h) = \int_Q \mathbf{a}(x, M_h u_h, D_x u_h) \cdot D_x u_h dxdt + \int_{Q_0} a_0(x, M_h u_h, D_x u_h) u_h dxdt = \\
&\int_Q \mathbf{a}(x, M_h u_h, D_x u_h) \cdot D_x u_h dxdt + \int_Q a_0(x, M_h u_h, D_x u_h) M_h u_h dxdt + \\
&\int_Q a_0(M_h u_h, D_x u_h)(u_h - M_h u_h) dxdt \geqslant C\int_Q |D_x u_h|^2 dxdt - \\
&|\int_Q a_0(x, M_h u_h, D_x u_h)(u_h - M_h u_h) dxdt| \geqslant C\int_Q |D_x u_h|^2 dxdt - \\
&C_2 h \int_Q |D_x u_h|^2 dxdt - C_0 = (C - C_2 h)\int_Q |D_x u_h|^2 dxdt - C_0.
\end{aligned} \tag{6.8}$$

Here we have used the fact that $|u_h - M_h u_h| < Ch|D_x u_h|$ in every triangular element $K \in \Pi_h$.

Q.E.D.

It can be easily shown that A_h is continuous which along with the coercivity guarantees the existence of the discrete solutions [**?**]. Moreover, because of the coerciveness we have the following uniform bound

$$\|u_h\|_{X^2} \leqslant C, \tag{6.9}$$

where u_h are solutions of (6.3). As a consequence, $u_h \to u$ weakly in X^2 (along a subsequence) as $h \to 0$. For further analysis the sequence u_h is fixed. It can be easily shown using the estimates for u_h that $A_h u_h$ is uniformly bounded in X^{-2} thus $A_h u_h \to g$ weakly in X^{-2}.

Next one can repeat the analysis in the proof of the Theorem 4.1. of [**?**] and obtain that

$$D_t u + g = f \text{ in } X^{-2}. \tag{6.10}$$

The crucial point is as in the Theorem 4.1 of [**?**] to prove $Au = g$. We would like to note that if the time discretization is taken into account, i.e.,

$$\frac{1}{\Delta t}(u_h(t) - u_h(t-\Delta t)) + A_h u_h = f_h,$$

(understood in a usual weak sense) then one can repeat the analysis of the Theorem 5.1. (Bardos-Brezis) in [**?**], and obtain again (6.10) with $D_t u$ replaced by Lu, where L is a corresponding generator. Thus the crucial point remains to show $Au = g$. We will not repeat these analyses here since they are identical. Thus our goal is to prove that $Au = g$. The next lemma is important for the proof of Theorem 6.1.

Lemma 6.3. *Let A_h and A be defined by* (6.4) *and* (6.6) *respectively. Then*

$$(A_h u_h - A u_h, v_h) \to 0,$$

for any uniformly bounded family of u_h and compact family of v_h in X^2. Moreover, if u_h is uniformly bounded in $X^{2+\alpha}$ ($\alpha > 0$) then

$$(A_h u_h - A u_h, u_h) \to 0. \tag{6.11}$$

Proof. Consider

$$\begin{aligned}(A_h u_h - A u_h, v_h) = \int_Q (&\mathbf{a}(x, M_h u_h, D_x u_h) - \mathbf{a}(x, u_h, D_x u_h)) \cdot D_x v_h + \\ &(a_0(x, M_h u_h, D_x u_h) - a_0(x, u_h, D_x u_h)) v_h) dxdt.\end{aligned} \tag{6.12}$$

Using the estimates for $\mathbf{a}$ we have

$$\begin{aligned}&|\int_Q (\mathbf{a}(x, M_h u_h, D_x u_h) - \mathbf{a}(x, u_h, D_x u_h)) \cdot D_x v_h dxdt| \leqslant \\ &C \int_Q (1 + |M_h u_h| + |D_x u_h| + |u_h|) \nu(|M_h u_h - u_h|) |D_x v_h| dxdt \leqslant \\ &C (\int_Q (1 + |u_h|^2 + |D_x u_h|^2) dxdt)^{1/2} (\int_K |D_x v_h|^2 \nu(|M_h u_h - u_h|)^2 dxdt)^{1/2} \leqslant \\ &C (1 + \|u_h\|^2_{X^2})^{1/2} (\int_Q |D_x v_h|^2 \nu(h |D_x u_h|)^2 dx)^{1/2}\end{aligned} \tag{6.13}$$

Here we have used $|u_h - M_h u_h| \leqslant Ch|D_x u_h|$ in every triangular element $K \in \Pi_h$. Because of Lemma 6.1 we obtain that the right hand side of (6.13) converges to zero for any uniformly bounded family of $u_h \in X^2$ and compact family $v_h \in X^2$ as $h \to 0$. The estimate for a_0 can be obtained in a similar way,

$$|\int_Q (a_0(x, M_h u_h, D_x u_h) - a_0(x, u_h, D_x u_h), D_x v_h) dxdt| \leqslant (C + \|u_h\|^2_{X^2})^{1/2} (\int_Q |v_h|^2 \nu(h|D_x u_h|)^2 dx)^{1/2}. \tag{6.14}$$

Note that the right hand side of (6.14) converges to zero for any uniformly bounded family of $u_h \in X^2$ and $v_h \in X^2$. Indeed, the latter implies that v_h is uniformly bounded in $X^{2+\alpha}$ for some $\alpha > 0$. Thus applying Lemma 6.1 we obtain that the right hand side of (6.14) converges to zero for any uniformly bounded family of v_h in $X^{2+\alpha}$.

To show (6.11) we note that

$$(A_h u_h - A u_h, u_h) \leqslant C(1 + \|u_h\|^2_{X^2})^{1/2} (\int_Q |D_x u_h|^2 \nu(h|D_x u_h|)^2 dxdt)^{1/2}. \tag{6.15}$$

Since $D_x u_h$ is uniformly bounded in $Y^{2+\alpha}(Q)$, $\alpha > 0$ we obtain that the right hand side of (6.15) converges to zero according to Lemma 6.1.

Q.E.D.

Lemma 6.4. *For some $\alpha > 0$ we have*

$$\|u_h\|_{X^{2+\alpha}} \leqslant C$$

Proof. To prove this lemma we use the results of the previous section. Consider the operator,

$$A_h u = -div(\mathbf{a}(x, M_h u_h, D_x u))$$

and

$$f_h = f - a_0((x, M_h u_h, D_x u_h),$$

where u_h is a discrete solution of (6.3). Then A_h is strongly monotone with operator constants independent of h. Moreover, using (6.9) we get

$$\|a_0(x, M_h u_h, D_x u_h)\|_{L^2(Q)} \leqslant C.$$

Clearly u_h is a solution of $A_h u_h = f_h$ (i.e., $(A_h u_h, w_h) = (f_h, w_h)$, $\forall w_h \in E_h$) and thus by Theorem 5.1 we have

$$\|u_h\|_{X^{2+\alpha}} \leqslant C,$$

for some $\alpha > 0$.

Q.E.D.

Lemma 6.5. *$Au_h \to f$ weakly in X^{-2} as $h \to 0$, where u_h are solutions of (6.3).*

Proof.
For any $v \in X^2$ and $v_h \to v$ in X^2 we have

$$\lim_{h\to 0}(Au_h, v_h) = \lim_{h\to 0}(A_h u_h, v_h) + \lim_{h\to 0}(Au_h - A_h u_h, v_h) = (f, v)$$

Here we have used the Lemma 6.3.
Q.E.D.
Proof (Theorem 6.1).
Thus we have the following:

$$u_h \to u \text{ weakly in } X^2, \quad Au_h \to f \text{ weakly in } X^{-2}, \quad (Au_h, u_h) \to (f, u). \qquad (6.16)$$

Since the operator A is type M [**?**] this guarantees that $Au = f$, i.e., u is a solution. Moreover, since our differential operators is also type S_+ (see e.g., [**?**], page 3) we have $u_h \to u$ strongly X^2. This completes the proof of the theorem.
Q.E.D.
Remark. To carry out the proof of Theorem 6.1, Meyers type estimates were needed In particular, in order for (6.11) to hold we need extra integrability of discrete solutions. The condition (6.11) (or the same as the third condition in (6.16)) is necessary for the convergence of discrete solutions to the continuous ones (cf. [**?**], page 38).
Remark. In this paper we have assumed Meyers type estimates for approximate solutions of the linear heat equation. We were not able to find the proof of this fact in the literature and it is a subject of our future research.

7. ACKNOWLEDGMENTS

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