Meyers type estimates for approximate solutions of nonlinear parabolic equations and their applications

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Abstract — In this paper we obtain Meyers type \((L^{p+\varepsilon})\) estimates for approximate solutions of nonlinear parabolic equations. This research is motivated by a numerical homogenization of these type of equations [?]. Using derived estimates we show the convergence of numerical solutions obtained from numerical homogenization methods.

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1. INTRODUCTION

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Meyers type regularity estimates [?] provide extra integrability of the solutions of differential equations. This property of the solutions is found to be useful in many applications. In this paper our goal is to derive Meyers type estimates for the approximate (numerical) solutions. The need for such estimates arises, for example, in the numerical homogenization of nonlinear parabolic equations (see [?], page 255). Only extra integrability of the discrete solutions guarantees their convergence to the solution of continuous equations under suitable assumptions.

In this paper we are interested in obtaining Meyers type \((L^{p+\varepsilon})\) estimates for numerical solutions of nonlinear parabolic equations. Our basic tool is the technique presented in [?,?] for continuous operators. The starting point of our approach is the use of Meyers type estimates for linear heat equations. Further, using adequate contraction maps we extend these estimates to strongly monotone operators. In the next step, using a particular discrete solution we obtain Meyers type estimates for more general parabolic operators of the form

\[
D_t u - \text{div}(a(x,u,D_x u)) + a_0(x,u,D_x u) = f
\]  

(2.1)

with homogeneous Dirichlet boundary conditions under quite general assumptions.

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on $a$ and $a_0$. To obtain Meyers type estimates for (2.1) we use weaker assumptions than those imposed in [?].

In the paper we also consider an application of the obtained estimates to a particular discretization of (2.1) which arises in a numerical homogenization of such equations [?]. This discretization of (2.1) is different from the standard Galerkin discretization. In particular, two different discrete spaces are involved for approximation of $u$ and $D_xu$. This scenario can not be avoided in a numerical homogenization procedure because the solution of the local problems do not belong to the discrete spaces that are used for approximation of the homogenized solutions [?]. To obtain the convergence of the discrete solutions to a solution of (2.1) one needs Meyers type estimates.

The paper is organized as follows. In the next section we discuss preliminary results regarding linear equations. In the following section we obtain the estimates for monotone operators. Section 4 is devoted to the Meyers type estimates for equations (2.1). Finally we use these estimates to prove the convergence of a numerical scheme.

3. PRELIMINARIES

Denote $Q = [0, T] \times Q_0$ and

$$
X^0 = L^p(0, T, W_0^1, p(Q_0)), \quad X^{-p} = L^p(0, T, W^{-1, p}(Q_0))
$$

$$
Y^p = L^p(0, T, L^p(Q_0)), \quad W^p = \{u \in X^p, D_tu \in X^{-p}, u(t = 0) = 0\},
$$

(3.1)

where $\|u\|_{W^p} = \|u\|_{X^p} + \|D_tu\|_{X^{-p}}$. Throughout the paper $C$ (with or without sub- (or super-)script ) denotes a generic constant which independent of $h$, unless otherwise stated.

Let $E_h$ be a family of finite dimensional subspaces such that $\text{span}(\bigcup h E_h)$ is dense in $W_0^1, p(Q_0)$, $2 \leq p \leq p_0$ (some fixed $p_0$). Consider $u_h(t) \in E_h$, $u_h(t = 0) = 0$,

$$
\lambda (D_tu_h, v_h) + (D_xu_h, D_xv_h) = (f, v_h), \quad \forall v_h \in E_h,
$$

(3.2)

where $\lambda$ is a parameter. $(u, v)$ means $\int_{Q_0} uvdx$ and the duality pairing between $W_0^1, p(Q_0)$ and $W^{-1, q}(Q_0)$, and $1/p + 1/q = 1$. Our basic assumption is that for any $f \in X^{-p}$, for some $p$, $2 \leq p \leq p_0$, we have

$$
\|u_h\|_{X^p} \leq C\|f\|_{X^{-p}}.
$$

(3.3)

$C = C_{p, \lambda}$ depends on $p$ and $\lambda$, but not on $h$. One can assume (3.3) only for $p \in [2, s]$, for some $s > 2$. This result for $p = 2$ can be easily obtained and known (e.g., [?]). Moreover, $C_{2, \lambda} \leq 1$. In the paper we assume Meyers type estimates for linear heat equation. (3.3) and (3.2) implies that $\|u_h\|_{W^p} \leq C_{p, \lambda}\|f\|_{X^{-p}}$, but in general $C_{p, \lambda} > 1$.

Denote by $L^\lambda_h$ the linear operator that maps $f$ into $u_h$,

$$
L^\lambda_h f = u_h.
$$
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Inequality (3.3) implies that $L^\lambda_h$ is a bounded linear operator from $X^{-p}$ into $W^p$, as well as from $X^{-p}$ to $X^p$. Denote by $M_{p,\lambda,h}$ the norm of the operators $L^\lambda_h$ from $X^{-p}$ to $X^p$ and set

$$M_{p,\lambda} = \sup_h M_{p,\lambda,h}.$$ 

It can be easily shown that $M_{2,\lambda} \leq 1$.

**Theorem 3.1.** Let $p_0 > s > 2$, $2 \leq p \leq s$ and

$$\frac{1}{p} = \frac{1 - \varrho}{2} + \frac{\varphi}{s}.$$ 

Then

$$M_{p,\lambda} \leq (M_{s,\lambda})^\varphi. \tag{3.4}$$

**Proof.**

Introduce a family of operators $B^\lambda_h$,

$$B^\lambda_h = D \circ L^\lambda_h \circ \text{div}.$$ 

The operator $B^\lambda_h$ is a bounded linear operator that acts in the space $Y^p$. Next we estimate the norm of $B^\lambda_h$ in $Y^p$.

$$\|B^\lambda_h\|_{Y^p} = \sup_{\|u\|_{L^1(\Omega)} \leq 1} \|D \circ L^\lambda_h \circ \text{div}(u)\|_{Y^p} = \sup_{\|f\|_{X^{-s}} \leq 1} \|D \circ L^\lambda_h \circ f\|_{Y^p} =$$

$$\sup_{\|f\|_{X^{-s}} \leq 1} \|L^\lambda_h \circ f\|_{X^s} = \|L^\lambda_h\|_{X^s} \leq M_{s,\lambda,h}. \tag{3.5}$$

Similarly we can check that $\|B^\lambda_h\|_{Y^s} \leq M_{2,\lambda,h} \leq 1$. Next we apply the Riesz-Thorin interpolation theorem [?] on $B^\lambda_h$. Let $s > 2$, $2 \leq p \leq s$ and

$$\frac{1}{p} = \frac{1 - \varrho}{2} + \frac{\varphi}{s}.$$ 

Then using Riesz-Thorin interpolation theorem we have

$$\|B^\lambda_h\|_{Y^p} \leq (M_{s,\lambda})^\varphi. \tag{3.6}$$

Introduce $z$

$$(z, D_h v) = (f, v), \forall v \in X^2.$$ 

Since $(z, D_h v) \leq \|f\|_{X^{-p}} \|v\|_{X^p}$, $z$ is a continuous linear functional on a subspace of $Y^q$. By the Hahn-Banach theorem, $z$ can be extended to a continuous linear functional on $Y^q$ and $\|z\|_{Y^q} \leq \|f\|_{X^{-p}}$. Moreover, $\text{div}(z) = f$. 

Denote $u_h = L_{h, \lambda} f$. Then $D_t u_h = B_h^\lambda z$ and

$$
\|D_x u_h\|_{Y^p} \leq M_{s, \lambda}^0 \|z\|_{Y^p} \leq M_{s, \lambda}^0 \|f\|_{X^{-p}}.
$$

Thus, $u_h \in X^p$ and

$$
\|u_h\|_{X^p} \leq M_{s, \lambda}^0 \|f\|_{X^{-p}}.
$$

Q.E.D.

4. MONOTONE OPERATORS

Consider

$$
D_t u + A u = f,
$$

where

$$
A u = -\text{div}(a(x, t, D_x u))
$$

and assume that

- $a : Q \times \mathbb{R}^n \to \mathbb{R}^n$ be a Caratheodory function and $a(x, t, 0) = 0$ (the latter is for the sake of simplicity).

- 

$$
|a(x, t, \xi_1) - a(x, t, \xi_2)| \leq M|\xi_1 - \xi_2| 
$$

for all $(x, t) \in Q$ and $\xi_1, \xi_2 \in \mathbb{R}^n$.

- 

$$
(a(x, t, \xi_1) - a(x, t, \xi_2)) \cdot (\xi_1 - \xi_2) \geq m|\xi_1 - \xi_2|^2.
$$

$A$ is a strictly monotone continuous operator from $X^2$ to $X^{-2}$, and

$$
(Au - Av, u - v) \geq m\|u - v\|^2_{X^{-2}}.
$$

Hence, it is coercive. In fact $A$ maps $X^2$ continuously into $X^{-2}$.

$$
\|Au - Av\|_{X^{-2}} \leq M\|u - v\|_{X^2}.
$$

Consider an approximate problem. Find $u_h \in E_h$ such that

$$
(D_t u_h, v_h) + (A u_h, v_h) = (f, v_h), \ \forall v_h \in E_h.
$$

(4.3)

For the analysis we will need the following notation

$$
\lambda = \frac{m}{M^2}, \ \ k = (1 - \frac{m^2}{M^2})^{1/2}.
$$

Our goal is to prove the following theorem.
Theorem 4.1. Let $u_h$ be the solution of (4.3). Then

$$
\|u_h\|_{W^p} \leq C\|f\|_{X^{-p}}.
$$

(4.4)

For the proof of the theorem we will need some auxiliary facts. Consider the operator $\overline{A}$ defined by

$$
\overline{A}u = -\Delta u - \lambda A(u).
$$

Lemma 4.1.

$$
\|\overline{A}u_1 - \overline{A}u_2\|_{X^{-p}} \leq k\|u_1 - u_2\|_{X^p},
$$

i.e., $\overline{A} : X^p \to X^{-p}$ is Lipschitz continuous with Lipschitz constant $k, k < 1$.

Proof.

The flux corresponding to $\overline{A}$ is given by

$$
\overline{\varphi}(x,t,\xi) = \xi - \lambda a(x,t,\xi).
$$

Next we would like to derive the following estimate for $\overline{\varphi}(x,\xi)$,

$$
|\overline{\varphi}(x,t,\xi_1) - \overline{\varphi}(x,t,\xi_2)| \leq (1 - \frac{m^2}{M^2})^{1/2}|\xi_1 - \xi_2| = k|\xi_1 - \xi_2|.
$$

(4.5)

Indeed,

$$
|\overline{\varphi}(x,t,\xi_1) - \overline{\varphi}(x,t,\xi_2)|^2 = (\xi_1 - \xi_2) \cdot (\xi_1 - \xi_2) - 2\lambda(a(x,t,\xi_1) - a(x,t,\xi_2)) \cdot (\xi_1 - \xi_2) +
\lambda^2(a(x,t,\xi_1) - a(x,t,\xi_2)) \cdot (a(x,t,\xi_1) - a(x,t,\xi_2)) =
|\xi_1 - \xi_2|^2 - 2\lambda(a(x,t,\xi_1) - a(x,t,\xi_2)) \cdot (\xi_1 - \xi_2) + \lambda^2|a(x,t,\xi_1) - a(x,t,\xi_2)|^2.
$$

(4.6)

Assumptions (4.1) and (4.2) imply

$$
|\overline{\varphi}(x,t,\xi_1) - \overline{\varphi}(x,t,\xi_2)|^2 \leq |\xi_1 - \xi_2|^2 - 2\lambda m|\xi_1 - \xi_2|^2 + \lambda^2 M^2 |\xi_1 - \xi_2|^2 =
(1 - 2\lambda m + \lambda^2 M^2)|\xi_1 - \xi_2|^2 = k^2|\xi_1 - \xi_2|^2.
$$

(4.7)

The estimate (4.5) implies immediately that for any $u_1, u_2 \in W^p$ and $v \in W^q$ we have

$$
|\langle \overline{A}u_1 - \overline{A}u_2, v \rangle| = |\int_Q (\overline{\varphi}(x,t,D_x u_1) - \overline{\varphi}(x,t,D_x u_2)) \cdot D_x v dx| \leq
\|\overline{\varphi}(x,t,D_x u_1) - \overline{\varphi}(x,t,D_x u_2)\|_{L^p(Q)}\|D_x v\|_{L^q(Q)} \leq k\|D(u_1 - u_2)\|_{W^p}\|D_x v\|_{W^q}.
$$

(4.8)

This means that

$$
\|\overline{A}u_1 - \overline{A}u_2\|_{X^{-p}} \leq k\|u_1 - u_2\|_{X^p},
$$

i.e., $\overline{A} : W^p \to X^{-p}$ is Lipschitz continuous with Lipschitz constant $k, k < 1$. 

Q.E.D.

Now we define the operator $Q_h = Q_{h,f}$ (if $f$ is fixed in $W^{-1,p}(Q)$ for some $p \geq 2$), $Q_h : E_h \rightarrow E_h$, by the formula $(v_h \in E_h)$

$$Q_h v_h = L_h^1(\overline{Av}_h + \overline{\lambda} f)$$

If $u_h \in E_h$ is a fixed point of $Q_h$, then $u_h$ is the approximate solution of $D_t u + Au = f$. Indeed, from $L_h^1 (\overline{Av}_h + \overline{\lambda} f) = v_h$ it follows that $(\overline{Av}_h + \overline{\lambda} f, w_h) = \langle \lambda D_t v_h - \Delta v_h, w_h \rangle$ $(\forall w_h \in E_h)$, which is equivalent to $(-\lambda Av_h + \overline{\lambda} f, w_h) = \langle \lambda D_t v_h, w_h \rangle$. The latter implies that $v_h$ is the solution of $(D_t v_h, w_h) + (\overline{Av}_h, w_h) = (f, w_h)$. We consider $E_h$ with the norm induced from $W^p$.

**Lemma 4.2.** $Q_h$ is Lipschitz continuous with the Lipschitz constant $M_p k$ from $X^p$ to itself,

$$\|Q_h u_h - Q_h v_h\|_{X^p} \leq M_p k \|u_h - v_h\|_{X^p}.$$  

**Proof.** Indeed,

$$\|Q_h u_h - Q_h v_h\|_{X^p} = \|L_h^1(\overline{Au}_h - \overline{Av}_h)\|_{X^p} \leq M_p \|\overline{Au}_h - \overline{Av}_h\|_{X^{p'}} \leq M_p k \|u_h - v_h\|_{X^p}$$  

(4.9)

Note that $k < 1$.

Q.E.D.

**Proof.** (Theorem 4.1).

Inequality (3.4) implies that if $s$ is sufficiently close to 2, then $M_p$ is close to 1 for all $p \in [2, s]$, and $M_p k < 1$ for $p \in [2, s]$ with $s$ close to 2.

Next we take $f, g \in X^{-p}$. Let $u_h$ and $w_h$ be the approximate solutions of

$$Au = f, \quad Aw = g.$$  

Then $u_h$ and $w_h$ are fixed points of $Q_{h,f}$ and $Q_{h,g}$, respectively and we have

$$\|u_h - w_h\|_{X^p} = \|Q_{h,f} u_h - Q_{h,g} w_h\|_{X^p} \leq M_p k \|u_h - w_h\|_{X^p} + M_p \|u_h - w_h\|_{X^{p'}} \leq M_p k \|u - w\|_{X^p} + M_p \|f - g\|_{X^{p'}}.$$  

(4.10)

Hence,

$$\|u_h - w_h\|_{X^p} \leq \frac{\lambda M_p}{1 - M_p k} \|f - g\|_{X^{p'}}.$$  

Further using the fact that $(D_t u_h, v_h) = (f,v_h)$ and $(D_t w_h + Aw_h, v_h) = (g,v_h)$ we have

$$\|D_t u_h - D_t w_h\|_{X^{p'}} = \|f - g - Au_h + Aw_h\|_{X^{p'}} \leq \|f - g\|_{X^{p'}} + M \|u_h - w_h\|_{X^p} \leq (1 + \frac{\lambda MM_p}{1 - M_p k}) \|f - g\|_{X^{p'}}.$$  

(4.11)
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From these estimates we obtain

\[ \|u_h - w_h\|_{W^p} \leq C\|f - g\|_{X^{-p}}. \]

With \( g = 0 \) we have

\[ \|u_h\|_{W^p} \leq C\|f\|_{X^{-p}}. \]  (4.12)

This completes the proof of the theorem.
Q.E.D.

5. GENERAL NONLINEAR PARABOLIC OPERATORS

Consider

\[ D_t u + A u = f, \]

where

\[ A u = -\text{div}(a(x,t,u,D_x u)) + a_0(x,t,u,D_x u) \]

and assume

- \( a : Q \times R \times R^n \to R^n, a_0 : Q \times R \times R^n \to R^n \) are Caratheodory functions, and for simplicity we assume \( a(x,t,0,0) = 0, a_0(x,t,0,0) = 0. \)

- \( |a(x,t,\eta,\xi)| + |a_0(x,t,\eta,\xi)| \leq C(1 + |\eta| + |\xi|). \)  (5.1)

- \( |a(x,t,\eta,\xi_1) - a(x,t,\eta,\xi_2)| \leq M|\xi_1 - \xi_2|. \)  (5.2)

- \( (a(x,t,\eta,\xi_1) - a(x,t,\eta,\xi_2)) \cdot (\xi_1 - \xi_2) \geq m|\xi_1 - \xi_2|^2. \)  (5.3)

- \( a(x,t,\eta,\xi) \cdot \xi + a_0(x,t,\eta,\xi) \eta \geq \alpha|\xi|^2 - \beta, \) \( \alpha > 0, \beta \geq 0. \)  (5.4)

\( A : X^2 \to X^{-2} \) is a continuous pseudomonotone \([?]\) (and type \( S_+ [?]\)) coercive operator. Hence \( D_t u + A u = f, f \in X^{-2}(Q) \) has a solution in \( W^2 \) (not necessarily unique). Consider an approximate problem

\[ (D_t u_h, v_h) + (A u_h, v_h) = (f,v_h), \forall v_h \in E_h. \]  (5.5)

The approximate problem has a solution \( u_h \in E_h \) (not necessarily unique) and

\[ \|u_h\|_{W^2} \leq C, \forall h. \]  (5.6)

This follows from the coerciveness of the operator \( A. \)
Theorem 5.1. Let \( u_h \) be a solution of (5.5). Then
\[
\| u_h \|_{W^p} \lesssim C, \quad \forall \ h,
\]
p \in [2, s], s is close to 2.

Proof.
Introduce the operators
\[
A_h u = -\text{div} a(x, u_h, D_x u)
\]
and
\[
f_h = f - a_0(x, u_h, D_x u_h).
\]
\( A_h \) is strongly monotone operator with operator constants independent of \( h \) and the estimate (5.6) implies that
\[
\| a_0(x, u_h, D_x u_h) \| \leq C.
\]
Since \( Y^2 \subset X^{-p}, p \in [2, s] \), for some \( s \), we have
\[
\| f_h \|_{X^{-p}} \leq C
\]
umiformly, provided \( f \in X^{-p} \).

Clearly, \( u_h \) is an approximate solution of \( A_h u_h = f_h \) (i.e., \( (A_h u, w_h) = (f, w_h), \forall w_h \in E_h \)) and (4.4) imply that
\[
\| u_h \|_{W^p} \lesssim C, \quad \forall \ h,
\]
p \in [2, s], s is close to 2.
Q.E.D.

In the next section we will apply (5.7) to a numerical scheme.

6. AN APPLICATION

Consider the equation, \( u(t) \in W^2 \)
\[
D_t u - \text{div}(a(x, t, u, D_x u)) + a_0(x, t, u, D_x u) = f.
\]

(6.1)

Let \( a \) and \( a_0 \) satisfy the assumptions (5.1)-(5.4) and also the following assumption, for any \( \xi, \xi' \in \mathbb{R}^n \) and \( \eta, \eta' \in \mathbb{R} \)

\[
|a(x, t, \eta, \xi) - a(x, t, \eta', \xi')| + |a_0(x, t, \eta, \xi) - a_0(x, t, \eta', \xi')| 
\leq C(1 + |\eta| + |\eta'| + |\xi| + |\xi'|) \nu(|\xi - \xi'|) + C(1 + |\eta|^{1-s} + |\eta'|^{1-s} + |\xi|^{1-s} + |\xi'|^{1-s})|\xi - \xi'|^s,
\]

(6.2)
for all \((x,t) \in \mathbb{R}^{n+1}\), where \(0 < s < 1\), \(v(r)\) is continuity modulus (i.e., a nondecreasing continuous function on \([0, +\infty)\) such that \(v(0) = 0\), \(v(r) > 0\) if \(r > 0\), and \(v(r) = 1\) if \(r > 1\). Note that (6.1) with the imposed conditions (5.1)-(5.4) and (6.2) may have multiple solutions.

The equation (6.1) has a solution and in this section we will be interested in the approximation of this solution with the following discretization. Let \(E_h\) be given by

\[
E_h = \{ v_h \in C^0(\overline{Q}) : \text{the restriction } v_h \text{ is linear for each triangle } K \subset \Pi_h \text{ and } v_h = 0 \text{ on } \partial Q \},
\]

where \(\Pi_h\) is a standard triangular discretization of \(Q_0\), \(\text{diam}(K) \leq C h\). We seek a solution, \(u_h(t) \in E_h\), of (6.1) in \(E_h\) such that

\[
(D_t u_h, v_h) + (A_h u_h, v_h) = (f, v_h), \quad \forall v_h \in E_h, \tag{6.3}
\]

where

\[
(A_h u_h, v_h) = \int_{Q_0} a(x,t,M_h u_h,D_x u_h) \cdot D_x v_h \, dx + \int_{Q_0} a_0(x,t,M_h u_h,D_x u_h) v_h \, dx. \tag{6.4}
\]

Here \(M_h\) is an averaging operator over the each element \(K \subset \Pi_h\) defined as follows. For each \(u_h \in E_h\),

\[
M_h u_h = \sum_K 1_K \frac{1}{K} \int_K u_h \, dx, \tag{6.5}
\]

where \(1_K\) is an indicator function of \(K\). Moreover, for any \(\varphi \in L^p(Q)\), \(M_h \varphi \to \varphi\) in \(L^p(Q)\) (see e.g., [?]). Note that the discretization (6.3) can be more tractable for computational purposes since the spatial dependence is not present and the quadrature step can be easily implemented.

Define \(A u_h\) on \(E_h\) by

\[
(A u_h, v_h) = \int_{Q_0} a(x,t,u_h,D_x u_h) \cdot D_x v_h \, dx + \int_{Q_0} a_0(x,t,u_h,D_x u_h) v_h \, dx. \tag{6.6}
\]

**Theorem 6.1.** \(u_h\) converges to \(u\) in \(X^2\) as \(h \to 0\) along a subsequence, where \(u_h\) is a solution of (6.3) and \(u\) is a solution of (6.1).

The proof of the theorem will be carried out in the following way. First we will show the coercivity of the discrete operator, then the uniform boundedness of the solutions in \(X^2\) and \(A_h u_h\) in \(X^{-2}\). This will imply that \(u_h \to u\) and \(A_h u_h \to g\) weakly in the corresponding spaces. Then using standard technique for parabolic equations we obtain that \(D_t u + g = f\) in \(X^{-2}\). It remains to show that \(g = A u\). The analysis of the latter requires Meyers type estimates.

Next lemma will be used in the proof of Theorem 6.1.

**Lemma 6.1.** If \(u_k \to 0\) in \(L^r(Q)\) (\(1 < r < \infty\)) as \(k \to \infty\) then

\[
\int_Q v(u_k) |v_k|^p \, dx dt \to 0, \quad \text{as } k \to \infty
\]
for all $v_k$ either (1) compact in $L^p(Q)$ or (2) bounded in $L^{p+\alpha}(Q)$, $\alpha > 0$. Here $v(r)$ is continuity modulus defined previously (see (6.2)) and $1 < p < \infty$.

Proof. Since $u_k$ converges in $L'$ it converges in measure. Consequently, for any $\delta > 0$ there exists $Q_\delta$ and $k_0$ such that $\text{meas}(Q \setminus Q_\delta) < \delta$ and $v(u_k) < \delta$ in $Q_\delta$ for $k > k_0$. Thus

$$\int_Q v(u_k)|v_k|^p dx dt = \int_{Q_\delta} v(u_k)|v_k|^p dx dt + \int_{Q \setminus Q_\delta} v(u_k)|v_k|^p dx dt \leq C\delta + C \int_{Q \setminus Q_\delta} |v_k|^p dx dt.$$  \hspace{1cm} (6.7)

Next we use the fact that if (1) or (2) satisfies then the set $v_k$ has equi-absolute continuous norm [?] (i.e., for any $\varepsilon > 0$ there exists $\zeta > 0$ such that $\text{meas}(Q_\zeta) < \zeta$ implies $\|P_{Q_\zeta} v_k\|_p < \varepsilon$, where $P_D f = \{f(x), \text{ if } x \in D; 0 \text{ otherwise}$. Consequently, the second term on the right hand side of (6.7) converges to zero as $\delta \to 0$. Q.E.D.

To prove Theorem 6.1 we first show that $A_h$ is coercive,

Lemma 6.2. $A_h$ is coercive for sufficiently small $h$, i.e.,

$$(A_h u_h, u_h) \geq C\|u_h\|_{X^2} - C_0,$$

where $C$ and $C_0$ are generic constants independent of $h$.

Proof.

$$\begin{align*}
(A_h u_h, u_h) &= \int_Q a(x, M_h u_h, D_x u_h) \cdot D_x u_h dx dt + \int_Q a_0(x, M_h u_h, D_x u_h) u_h dx dt = \\
&= \int_Q a(x, M_h u_h, D_x u_h) \cdot D_x u_h dx dt + \int_Q a_0(x, M_h u_h, D_x u_h) M_h u_h dx dt + \\
&\geq C \int_Q |D_x u_h|^2 dx dt - C_2 h \int_Q |D_x u_h|^2 dx dt - C_0 = (C - C_2 h) \int_Q |D_x u_h|^2 dx dt - C_0.
\end{align*}$$  \hspace{1cm} (6.8)

Here we have used the fact that $|u_h - M_h u_h| < Ch|D_x u_h|$ in every triangular element $K \in \Pi_h$. Q.E.D.

It can be easily shown that $A_h$ is continuous which along with the coercivity guarantees the existence of the discrete solutions [?]. Moreover, because of the coerciveness we have the following uniform bound

$$\|u_h\|_{X^2} \leq C,$$  \hspace{1cm} (6.9)
where \( u_h \) are solutions of (6.3). As a consequence, \( u_h \rightarrow u \) weakly in \( X^2 \) (along a subsequence) as \( h \rightarrow 0 \). For further analysis the sequence \( u_h \) is fixed. It can be easily shown using the estimates for \( u_h \) that \( A_h u_h \) is uniformly bounded in \( X^{-2} \) thus \( A_h u_h \rightarrow g \) weakly in \( X^{-2} \).

Next one can repeat the analysis in the proof of the Theorem 4.1. of [?] and obtain that

\[
D_i u + g = f \text{ in } X^{-2}. \tag{6.10}
\]

The crucial point is as in the Theorem 4.1 of [?] to prove \( Au = g \). We would like to note that if the time discretization is taken into account, i.e.,

\[
\frac{1}{\Delta t}(u_h(t) - u_h(t - \Delta t)) + A_h u_h = f_h,
\]

(understood in a usual weak sense) then one can repeat the analysis of the Theorem 5.1. (Bardos-Brezis) in [?], and obtain again (6.10) with \( D_i u \) replaced by \( L u \), where \( L \) is a corresponding generator. Thus the crucial point remains to show \( Au = g \). We will not repeat these analyses here since they are identical. Thus our goal is to prove that \( Au = g \). The next lemma is important for the proof of Theorem 6.1.

**Lemma 6.3.** Let \( A_h \) and \( A \) be defined by (6.4) and (6.6) respectively. Then

\[
(A_h u_h - A u_h, v_h) \rightarrow 0,
\]

for any uniformly bounded family of \( u_h \) and compact family of \( v_h \) in \( X^2 \). Moreover, if \( u_h \) is uniformly bounded in \( X^{2+\alpha} \) (\( \alpha > 0 \)) then

\[
(A_h u_h - A u_h, u_h) \rightarrow 0. \tag{6.11}
\]

**Proof.** Consider

\[
(A_h u_h - A u_h, v_h) = \int_Q (\mathbf{a}(x, M_h u_h, D_x u_h) - \mathbf{a}(x, u_h, D_x u_h)) \cdot D_x v_h + (a_0(x, M_h u_h, D_x u_h) - a_0(x, u_h, D_x u_h)) v_h dxdt. \tag{6.12}
\]

Using the estimates for \( \mathbf{a} \) we have

\[
\left| \int_Q (\mathbf{a}(x, M_h u_h, D_x u_h) - \mathbf{a}(x, u_h, D_x u_h)) \cdot D_x v_h dxdt \right| \leq C \int_Q (1 + |M_h u_h| + |D_x u_h| + |u_h|) \nu(|M_h u_h - u_h|)|D_x v_h| dxdt \leq C(\int_Q (1 + |u_h|^2 + |D_x u_h|^2)dxdt)^{1/2} (\int_K |D_x v_h|^2 \nu(|M_h u_h - u_h|)^2 dxdt)^{1/2} \leq \]

\[
C(1 + \|u_h\|_{X^2}^2)^{1/2} (\int_Q |D_x v_h|^2 \nu(h|D_x u_h|)^2 dx)^{1/2}
\]

where \( \nu \) and \( K \) are measurable in \( Q \).
Here we have used $|u_h - M_h u_h| \leq C h |D_x u_h|$ in every triangular element $K \in \Pi_h$. Because of Lemma 6.1 we obtain that the right hand side of (6.13) converges to zero for any uniformly bounded family of $u_h \in X^2$ and compact family $v_h \in X^2$ as $h \to 0$. The estimate for $a_0$ can be obtained in a similar way,

$$ \left| \int_Q (a_0(x, M_h u_h, D_x u_h) - a_0(x, u_h, D_x u_h), D_x v_h) dx dt \right| \leq (C + \|u_h\|_{X^2}^2)^{1/2} \left( \int_Q |v_h|^2 v(h|D_x u_h|)^2 dx \right)^{1/2}. $$

(6.14)

Note that the right hand side of (6.14) converges to zero for any uniformly bounded family of $u_h \in X^2$ and $v_h \in X^2$. Indeed, the latter implies that $v_h$ is uniformly bounded in $X^2 + a$ for some $a > 0$. Thus applying Lemma 6.1 we obtain that the right hand side of (6.14) converges to zero for any uniformly bounded family of $v_h$ in $X^2 + a$.

To show (6.11) we note that

$$(A_h u_h - A u_h, u_h) \leq C (1 + \|u_h\|_{X^2}^2)^{1/2} \left( \int_Q |D_x u_h|^2 v(h|D_x u_h|)^2 dx dt \right)^{1/2}. $$

(6.15)

Since $D_x u_h$ is uniformly bounded in $Y^{2+\alpha}(Q)$, $\alpha > 0$ we obtain that the right hand side of (6.15) converges to zero according to Lemma 6.1.

Q.E.D.

**Lemma 6.4.** For some $\alpha > 0$ we have

$$ \|u_h\|_{X^{2+\alpha}} \leq C $$

**Proof.** To prove this lemma we use the results of the previous section. Consider the operator,

$$ A_h u = -\text{div}(a(x, M_h u_h, D_x u)) $$

and

$$ f_h = f - a_0((x, M_h u_h, D_x u_h), $$

where $u_h$ is a discrete solution of (6.3). Then $A_h$ is strongly monotone with operator constants independent of $h$. Moreover, using (6.9) we get

$$ \|a_0(x, M_h u_h, D_x u_h)\|_{L^2(Q)} \leq C. $$

Clearly $u_h$ is a solution of $A_h u_h = f_h$ (i.e., $(A_h u_h, w_h) = (f_h, w_h), \forall w_h \in E_h)$ and thus by Theorem 5.1 we have

$$ \|u_h\|_{X^{2+\alpha}} \leq C, $$

for some $\alpha > 0$.

Q.E.D.

**Lemma 6.5.** $A u_h \to f$ weakly in $X^{-2}$ as $h \to 0$, where $u_h$ are solutions of (6.3).
Proof.
For any \( v \in X^2 \) and \( v_h \to v \) in \( X^2 \) we have
\[
\lim_{h \to 0} (Au_h,v_h) = \lim_{h \to 0} (A_h u_h,v_h) + \lim_{h \to 0} (Au_h - A_h u_h,v_h) = (f,v)
\]
Here we have used the Lemma 6.3.
Q.E.D.

Proof (Theorem 6.1).
Thus we have the following:
\[
\begin{align*}
& u_h \to u \text{ weakly in } X^2, \quad Au_h \to f \text{ weakly in } X^{-2}, \quad (Au_h,u_h) \to (f,u).
\end{align*}
\]
(6.16)
Since the operator \( A \) is type \( M \) this guarantees that \( Au = f \), i.e., \( u \) is a solution. Moreover, since our differential operators is also type \( S_+ \) (see e.g., [?], page 3) we have \( u_h \to u \) strongly \( X^2 \). This completes the proof of the theorem.
Q.E.D.

Remark. To carry out the proof of Theorem 6.1, Meyers type estimates were needed. In particular, in order for (6.11) to hold we need extra integrability of discrete solutions. The condition (6.11) (or the same as the third condition in (6.16)) is necessary for the convergence of discrete solutions to the continuous ones (cf. [?], page 38).

Remark. In this paper we have assumed Meyers type estimates for approximate solutions of the linear heat equation. We were not able to find the proof of this fact in the literature and it is a subject of our future research.

7. ACKNOWLEDGMENTS

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