Math 409. Fall 07 Name: ______________________

First Midterm

Instructions: *Show all of your work. Answers without sufficient justification will receive little or no credit.*

1. *(25 points) Consider the set $B \subset [0,1]$ given by*

$$B = \{x : x = \sum_{k=1}^{\infty} \frac{d_k}{10^k}, d_k \in \{0,1\}\}.$$

Decide whether B is countable or not. Fully prove and justify your answer.

Solution *B is UNCOUNTABLE since no function $f : N \longrightarrow B$ can be onto. To prove that we use Cantor's diagonalization argument as follows. Let f be such a function. Write*

$$f(n) = \sum_{k=1}^{\infty} \frac{d_{n,k}}{10^k}$$

Set e_k to be 1 if $d_{k,k}$ is 0 and 0 if $d_{k,k}$ is 1. Then $\alpha = \sum_{k=1}^{\infty} \frac{e_k}{10^k}$ is an element of B that is not equal to $f(n)$ for any natural number n (as they differ in the nth decimal), and hence f is not onto.

2. *(25 points) Let x_n be a sequence that satisfies*

$$|x_{n+1} - x_n| \le \frac{1}{n} - \frac{1}{n+1}, \quad \forall n \ge 1.$$

Decide whether it converges or diverges. Give full details of your deduction.

Solution *We know a sequence is convergent if and only if it is Cauchy. We study the later as it is more suitable to the problem.*

Note that for $m \ge n+1$

$$\begin{aligned} |x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+2} - x_{n+1}) + (x_{n+1} - x_n)| \\ &\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n| \\ &\le \frac{1}{m-1} - \frac{1}{m} + \frac{1}{m-2} - \frac{1}{m-1} + \cdots + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n} - \frac{1}{n+1} \\ &= \frac{1}{n} - \frac{1}{m} < 1/n \end{aligned}$$

(This can be written in summation notation too, which is more compact and precise)

$$|x_m - x_n| = |\sum_{k=n}^{m-1} x_{k+1} - x_k| \le \sum_{k=n}^{m-1} |x_{k+1} - x_k| \le \sum_{k=n}^{m-1} \frac{1}{k} - \frac{1}{k+1} = \frac{1}{n} - \frac{1}{m} < 1/n$$

So, Given $\epsilon > 0$, choose n_0 such that $\frac{1}{n_0} < \epsilon$, then for $m \ge n \ge n_0$, we have

$$|x_m - x_n| < 1/n \le 1/n_0 < \epsilon.$$

So the sequence is Cauchy and hence converges.

3. (a) *(20 points) Let x_n be a bounded sequence. Prove (directly, without referring to any theorems) that if x_n has a convergent subsequence and is increasing, then in fact the whole sequence must be convergent.*

Solution *Write the converging subsequence as $\{x_{n_k}\}_{k=1}^{\infty}$ and its limit as a. Given $\epsilon > 0$, there exists a k_0 such that*

$$|x_{n_k} - a| < \epsilon$$

whenever $k \geq k_0$. Also, note that we must have $a \geq x_{n_k}$ for all k (Since an increaing sequence can't converge to something less than its supremum; see below for a short proof of this fact). Now take n_0 to be n_{k_0}. For $n \geq n_0$ we can find an $l \geq n_0$ such that $n \leq n_l$. Since the sequence is increasing we have

$$x_{n_0} \leq x_n \leq x_{n_l} \leq a.$$

Hence

$$|a - x_n| = a - x_n \leq a - x_{n_0} < \epsilon.$$

It follows that x_n converges to a.
(Lastly we show that we must have had $a \geq x_{n_k}$ for all k without referring to suprema. Otherwise, if say $a < x_{n_l}$ for some l, then taking $\epsilon_1 = x_{n_l} - a > 0$, then for all $k \geq l$, we have

$$|x_{n_k} - a| = (x_{n_k} - x_{n_l}) + (x_{n_l} - a) = \epsilon_1 + (x_{n_k} - x_{n_l}),$$

where we could discard the absolute value symbol as its argument is positive.)

(b) *(5 points) The above statement is equivalent to stating that "Theorem X implies Theorem Y", where X and Y are theorems we studied. What are they?*

X=Bolzano-Weierstrass Theorem: Every bounded sequence has a convergent subsequence

Y=Monotone Convergence Theorem: Every monotone bounded sequence converges.

4. *(25 points) Clearly define the following principles (See the text for these defintions).*

   (a) *The well ordering principle*

   (b) *The binomial formula*

   (c) *The principle of mathematical induction*

   (d) *A sequence that diverges to* $+\infty$

   (e) *The nested interval property of real numbers*