Lectures on the $\mathcal{L}^2$-Sobolev Theory of the $\overline{\partial}$-Neumann Problem

Emil J. Straube

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Preface

In the summer and fall of 2005, I gave a series of lectures at the Erwin Schrödinger International Institute of Mathematical Physics in Vienna on the basic $\mathcal{L}^2$-Sobolev theory of the $\overline{\partial}$-Neumann problem on domains in $\mathbb{C}^n$. These lectures, aimed at advanced graduate students and young researchers, produced a set of notes of about eighty pages. In the spring semester of 2006, I taught a graduate course at Texas A&M University on the same topic, revising and somewhat expanding the notes in the process. Subsequently, I set out to further revise the notes and to make them more or less self-contained, as far as the $\overline{\partial}$-Neumann problem is concerned. The intent remained the same: to provide a thorough introduction to the topic in the title, in the setting of bounded pseudoconvex domains in $\mathbb{C}^n$, that leads up to current research. This monograph is the result.

Chapter 2 presents the basic $\mathcal{L}^2$-theory. Chapter 3 discusses the subelliptic estimates on strictly pseudoconvex domains. From the point of view of leading up to current research, chapter 4 on compactness and chapter 5 on regularity in Sobolev spaces are the most important. For a detailed description of the contents of these chapters, along with historical remarks, I refer the reader to the introductory chapter 1.

A word about prerequisites. The reader is assumed to have a solid background in basic complex and functional analysis (including the elementary $\mathcal{L}^2$-Sobolev theory and distributions). Some knowledge in several complex variables is clearly helpful, if only for motivation. Concerning partial differential equations, not much is assumed. The elliptic regularity of the Dirichlet problem for the Laplacian is quoted a few times. On the other hand, the ellipticity results needed for elliptic regularization in chapter 3 are proved from scratch.

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I was involved in some of the results presented here. For the most part, these results were obtained through joint work. I would like to take this opportunity to thank my coauthors over the years for sharing their ideas.

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