Finite Mathematics

Applications and Technology

First Edition

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Finite Mathematics: Applications and Technology, First Edition
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CHAPTER 1

Systems of Linear Equations and Models

CONNECTION

Demand for Televisions

As sleek flat-panel and high-definition television sets became more affordable, sales soared during the holidays. Sales of ultra-thin, wall-mountable LCD TVs rose over 100% in 2005 to about 20 million sets while plasma-TV sales rose at a similar pace, to about 5 million sets. Normally set makers and retailers lower their prices after the holidays, but since there was strong demand and production shortages for these sets, prices were kept high.

Source: http://biz.yahoo.com

1.1 Mathematical Models

APPLICATION
Cost, Revenue, and Profit Models

A firm has weekly fixed costs of \$80,000 associated with the manufacture of dresses that cost \$25 per dress to produce. The firm sells all the dresses it produces at \$75 per dress. Find the cost, revenue, and profit equations if x is the number of dresses produced per week. See Example 3 for the answer.

We will first review some basic material on functions. An introduction to the mathematical theory of the business firm with some necessary economics background is provided. We study mathematical business models of cost, revenue, profit, and depreciation, and mathematical economic models of demand and supply.

HISTORICAL NOTE

Augustin Cournot (1801–1877)

The first significant work dealing with the application of mathematics to economics was Cournot's *Researches into the Mathematical Principles of the Theory of Wealth,* published in 1836. It was Cournot who originated the supply and demand curves that are discussed in this section. Irving Fisher, a prominent economics professor at Yale University and one of the first exponents of mathematical economics in the United States, wrote that Cournot's book "seemed a failure when first published. It was far in advance of the times. Its methods were too strange, its reasoning too intricate for the crude and confident notions of political economy then current."

✧ Functions

Mathematical modeling is an attempt to describe some part of the real world in mathematical terms. Our models will be functions that show the relationship between two or more variables. These variables will represent quantities that we wish to understand or describe. Examples include the price of gasoline, the cost of producing cereal or the number of video games sold. The idea of representing these quantities as variables in a function is central to our goal of creating models to describe their behavior. We will begin by reviewing the concept of functions. In short, we call any rule that assigns or corresponds to each element in one set precisely one element in another set a function.

For example, suppose you are going a steady speed of 40 miles per hour in a car. In one hour you will travel 40 miles; in two hours you will travel 80 miles; and so on. The distance you travel depends on (corresponds to) the time. Indeed, the equation relating the variables distance (d), velocity (v), and time (t), is $d = v \cdot t$. In our example, we have a constant velocity of $v = 40$, so $d = 40 \cdot t$. We can view this as a correspondence or rule: Given the time t in hours, the rule gives a distance d in miles according to $d = 40 \cdot t$. Thus, given $t = 3$, $d = 40 \cdot 3 = 120$. Notice carefully how this rule is **unambiguous**. That is, given any time t, the rule specifies one and only one distance d. This rule is therefore a function; the correspondence is between time and distance.

Often the letter f is used to denote a function. Thus, using the previous example, we can write $d = f(t) = 40 \cdot t$. The symbol $f(t)$ is read "f of t." One can think of the variable t as the "input" and the value of the variable $d = f(t)$ as the "output." For example, an input of $t = 4$ results in an output of $d = f(4) = 40 \cdot 4 = 160$ miles. The following gives a general definition of a function.

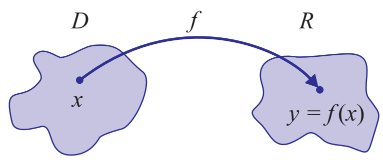


Figure 1.1
A function as a mapping

Definition of a Function
A **function** f from D to R is a rule that assigns to each element x in D one and only one element $y = f(x)$ in R. See Figure 1.1.

The set D in the definition is called the **domain** of f. We might think of the domain as the set of inputs. We then can think of the values $f(x)$ as outputs. The set of outputs, R is called the **range** of f.

Another helpful way to think of a function is shown in Figure 1.2. Here the function f accepts the input x from the conveyor belt, operates on x, and outputs (assigns) the new value $f(x)$.

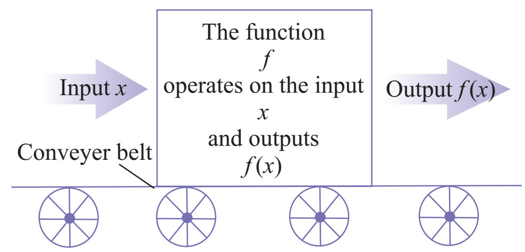


Figure 1.2
A function as a process

The letter representing elements in the domain is called the **independent variable** and the letter representing the elements in the range is called the **dependent variable**. Thus, if $y = f(x)$, x is the independent variable, and y is the dependent variable, since the value of y **depends** on x. In the equation $d = 40t$, we can write $d = f(t) = 40t$ with t as the independent variable. The dependent variable is d, since the distance **depends** on the spent time t traveling. We are free to set the independent variable t equal to any number of values in the domain. The domain for this function is $t \geq 0$ since only nonnegative time is allowed.

REMARK: The domain in an application problem will always be those values that are allowed for the independent variable in the particular application. This often means that we are restricted to non-negative values or perhaps we will be limited to the case of whole numbers only, as in the next example.

EXAMPLE 1 **Steak Specials** A restaurant serves a steak special for \$12. Write a function that models the amount of revenue made from selling these specials. How much revenue will 10 steak specials earn?

Solution We first need to decide if the independent variable is the price of the steak specials, the number of specials sold, or the amount of revenue earned. Since the price is fixed at \$12 per special and revenue depends on the number of specials sold, we choose the independent variable, x, to be the number of specials sold and the dependent variable, $R = f(x)$ to be the amount of revenue. Our rule will be $R = f(x) = 12x$, where x is the number of steak specials sold and R is the revenue from selling these

Technology Option
Example 1 is solved using a graphing calculator in Technology Note 1 on page 12.

specials in dollars. Note that x must be a whole number, so the domain is $x = 0, 1, 2, 3, \ldots$. To determine the revenue made on selling 10 steak specials, plug $x = 10$ into the model:

$$R = f(10) = 12(10) = 120$$

So the revenue is \$120. ✦

Recall that lines satisfy the equation $y = mx + b$. Actually, we can view this as a **function**. We can set $y = f(x) = mx + b$. Given any number x, $f(x)$ is obtained by multiplying x by m and adding b. More specifically, we call the function $y = f(x) = mx + b$ a **linear function**.

Definition of Linear Function
A **linear function** f is any function of the form,

$$y = f(x) = mx + b$$

where m and b are constants.

EXAMPLE 2 **Linear Functions** Which of the following functions are linear?

a. $y = -0.5x + 12$

b. $5y - 2x = 10$

c. $y = 1/x + 2$

d. $y = x^2$

Technology Option
You can graph the functions on a calculator to verify your results. Linear functions will be a straight line in any size window.

Solution

a. This is a linear function. The slope is $m = -0.5$ and the y-intercept is $b = 12$.

b. Rewrite this function first as,

$$\begin{aligned} 5y - 2x &= 10 \\ 5y &= 2x + 10 \\ y &= (2/5)x + 2 \end{aligned}$$

Now we see it is a linear function with $m = 2/5$ and $b = 2$.

c. This is not a linear function. Rewrite $1/x$ as x^{-1} and this shows that we do not have a term mx and so this is not a linear function.

d. x is raised to the second power and so this is not a linear function. ✦

✧ Mathematical Modeling

When we use mathematical modeling we are attempting to describe some part of the real world in mathematical terms, just as we have done for the distance traveled and the revenue from selling meals. There are three steps in mathematical modeling: formulation, mathematical manipulation, and evaluation.

Formulation

First, on the basis of observations, we must state a question or formulate a hypothesis. If the question or hypothesis is too vague, we need to make it precise. If it is too ambitious, we need to restrict it or subdivide it into manageable parts. Second, we need to identify important factors. We must decide which quantities and relationships are important to answer the question and which can be ignored. We then need to formulate a **mathematical** description. For example, each important quantity should be represented by a variable. Each relationship should be represented by an equation, inequality, or other mathematical construct. If we obtain a function, say, $y = f(x)$, we must carefully identify the input variable x and the output variable y and the units for each. We should also indicate the interval of values of the input variable for which the model is justified.

Mathematical Manipulation

After the mathematical formulation, we then need to do some mathematical manipulation to obtain the answer to our original question. We might need to do a calculation, solve an equation, or prove a theorem. Sometimes the mathematical formulation gives us a mathematical problem that is impossible to solve. In such a case, we will need to reformulate the question in a less ambitious manner.

Evaluation

Naturally, we need to check the answers given by the model with real data. We normally expect the mathematical model to describe only a very limited aspect of the world and to give only approximate answers. If the answers are wrong or not accurate enough for our purposes, then we will need to identify the sources of the model's shortcomings. Perhaps we need to change the model entirely, or perhaps we need to just make some refinements. In any case, this requires a new mathematical manipulation and evaluation. Thus, modeling often involves repeating the three steps of formulation, mathematical manipulation, and evaluation.

We will next create linear mathematical models by finding equations that relate cost, revenue, and profits of a manufacturing firm to the number of units produced and sold.

✧ Cost, Revenue, and Profit

Any manufacturing firm has two types of costs: fixed and variable. **Fixed costs** are those that do not depend on the amount of production. These costs include real estate taxes, interest on loans, some management salaries, certain minimal maintenance, and protection of plant and equipment. **Variable costs** depend on the amount of production. They include the cost of

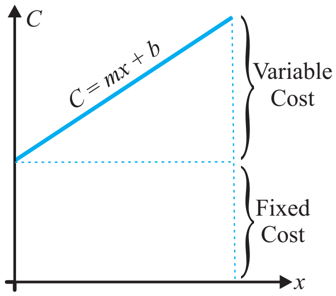


Figure 1.3
A linear cost function

material and labor. Total cost, or simply **cost**, is the sum of fixed and variable costs:

$$\text{cost} = \text{variable cost} + \text{fixed cost}$$

Let x denote the number of units of a given product or commodity produced by a firm. (Notice that we must have $x \geq 0$.) The units could be bales of cotton, tons of fertilizer, or number of automobiles. In the **linear cost model** we assume that the cost m of manufacturing one unit is the same no matter how many units are produced. Thus, the variable cost is the number of units produced times the cost of each unit:

$$\begin{aligned} \text{variable cost} &= (\text{cost per unit}) \times (\text{number of units produced}) \\ &= mx \end{aligned}$$

If b is the fixed cost and $C(x)$ is the cost, then we have the following:

$$\begin{aligned} C(x) &= \text{cost} \\ &= (\text{variable cost}) + (\text{fixed cost}) \\ &= mx + b \end{aligned}$$

Notice that we must have $C(x) \geq 0$. In the graph shown in Figure 1.3, we see that the y-intercept is the fixed cost and the slope is the cost per item.

CONNECTION
What Are Costs?

Isn't it obvious what the costs to a firm are? Apparently not. On July 15, 2002, Coca-Cola Company announced that it would begin treating stock-option compensation as a cost, thereby lowering earnings. If all companies in the Standard and Poor's 500 stock index were to do the same, the earnings for this index would drop by 23%.
Source: The Wall Street Journal, July 16, 2002

In the **linear revenue model** we assume that the price p of a unit sold by a firm is the same no matter how many units are sold. (This is a reasonable assumption if the number of units sold by the firm is small in comparison to the total number sold by the entire industry.) Revenue is always the price per unit times the number of units sold. Let x be the number of units sold. For convenience, we always assume that **the number of units sold equals the number of units produced**. Then, if we denote the revenue by $R(x)$,

$$\begin{aligned} R(x) &= \text{revenue} \\ &= (\text{price per unit}) \times (\text{number sold}) \\ &= px \end{aligned}$$

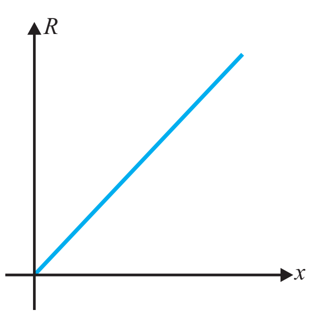


Figure 1.4
A linear revenue function

Since $p > 0$, we must have $R(x) \geq 0$. Notice in Figure 1.4 that the straight line goes through $(0, 0)$ because nothing sold results in no revenue. The slope is the price per unit.

CONNECTION
What Are Revenues?

The accounting practices of many telecommunications companies, such as Cisco and Lucent, have been criticized for what the companies consider revenues. In particular, these companies have loaned money to other companies, which then use the proceeds of the loan to buy telecommunications equipment from Cisco and Lucent. Cisco and Lucent then book these sales as "revenue." But is this revenue?

Regardless of whether our models of cost and revenue are linear or not, **profit** P is always revenue less cost. Thus

$$\begin{aligned} P &= \text{profit} \\ &= (\text{revenue}) - (\text{cost}) \\ &= R - C \end{aligned}$$

Recall that both cost $C(x)$ and revenue $R(x)$ must be nonnegative functions. However, the profit $P(x)$ can be positive or negative. Negative profits are called **losses**.

Let's now determine the cost, revenue, and profit equations for a dress-manufacturing firm.

EXAMPLE 3 **Cost, Revenue, and Profit Equations** A firm has weekly fixed costs of \$80,000 associated with the manufacture of dresses that cost \$25 per dress to produce. The firm sells all the dresses it produces at \$75 per dress.

a. Find the cost, revenue, and profit equations if x is the number of dresses produced per week.

b. Make a table of values for cost, revenue, and profit for production levels of 1000, 1500, and 2000 dresses and discuss what is the table means.

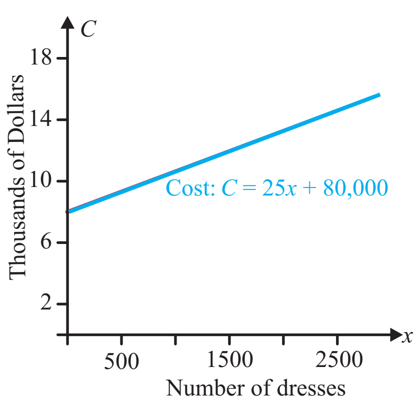


Figure 1.5a

Solution

a. The fixed cost is \$80,000 and the variable cost is $25x$. So

$$\begin{aligned} C &= (\text{variable cost}) + (\text{fixed cost}) \\ &= mx + b \\ &= 25x + 80,000 \end{aligned}$$

See Figure 1.5a. Notice that $x \geq 0$ and $C(x) \geq 0$.

b. The revenue is just the price \$75 that each dress is sold for multiplied by the number x of dresses sold. So

$$\begin{aligned} R &= (\text{price per dress}) \times (\text{number sold}) \\ &= px \\ &= 75x \end{aligned}$$

See Figure 1.5b. Notice that $x \geq 0$ and $R(x) \geq 0$. Also notice that if there are no sales, then there is no revenue, that is, $R(0) = 0$.

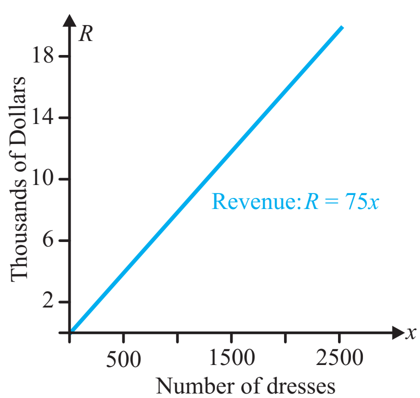


Figure 1.5b

Profit is always revenue less cost. So

$$\begin{aligned} P &= (\text{revenue}) - (\text{cost}) \\ &= R - C \\ &= (75x) - (25x + 80,000) \\ &= 50x - 80,000 \end{aligned}$$

See Figure 1.5c. Notice in Figure 1.5c that profits can be negative.

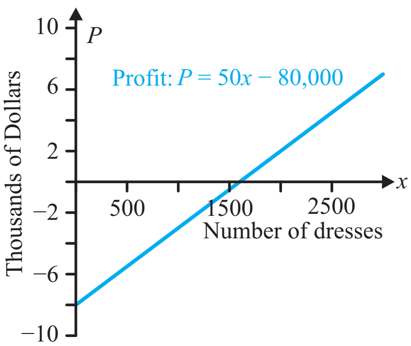


Figure 1.5c

c. When 1000 dresses are produced and sold $x = 1000$ so we have

$$\begin{aligned} C(1000) &= 25(1000) + 80,000 = 105,000 \\ R(1000) &= 75(1000) = 75,000 \\ P(1000) &= 75,000 - 105,000 = -30,000 \end{aligned}$$

Thus, if 1000 dresses are produced and sold, the cost is \$105,000, the revenue is \$75,000, and there is a negative profit or **loss** of \$30,000. Doing the same for 1500 and 2000 dresses, we have the results shown in Table 1.1.

Number of Dresses Made and Sold	1000	1500	2000
Cost in dollars	105,000	117,500	130,000
Revenue in dollars	75,000	112,500	150,000
Profit (or loss) in dollars	−30,000	−5,000	20,000

Table 1.1

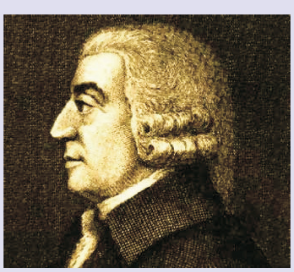
We can see in Figure 1.5c or in Table 1.1, that for smaller values of x, $P(x)$ is **negative**; that is, the firm has losses as their costs are greater than their revenue. For larger values of x, $P(x)$ turns positive and the firm has (positive) profits. ✦

✧ Supply and Demand

In the previous discussion we assumed that the number of units produced and sold by the given firm was small in comparison to the number sold by the industry. Under this assumption it was reasonable to conclude that the price, p, was constant and did not vary with the number x sold. But if the number of units sold by the firm represented a **large** percentage of the number sold by the entire industry, then trying to sell significantly more units could only be accomplished by **lowering** the price of each unit. Since we just stated that the price effects the number sold, you would expect the price to be the independent variable and thus graphed on the horizontal axis. However, by custom, the price is graphed on the vertical axis and the quantity x on the horizontal axis. This convention was started by English economist Alfred Marshall (1842–1924) in his important book, *Principles of Economics*. We will abide by this custom in this text.

For most items the relationship between quantity and price is a decreasing function (there are some exceptions to this rule, such as certain luxury goods, medical care, and higher eduction, to name a few). That is, for the

HISTORICAL NOTE



Adam Smith (1723–1790)

Adam Smith was a Scottish political economist. His *Inquiry into the Nature and Causes of the Wealth of Nations* was one of the earliest attempts to study the development of industry and commerce in Europe. That work helped to create the modern academic discipline of economics. In the Western world, it is arguably the most influential book on the subject ever published.

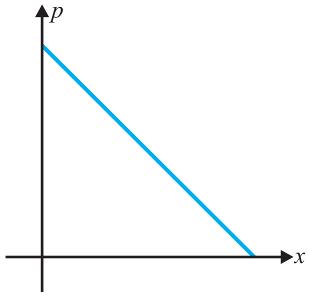


Figure 1.6
A linear demand function

number of items to be sold to increase, the price must decrease. We assume now for mathematical convenience that this relationship is linear. Then the graph of this equation is a straight line that slopes downward as shown in Figure 1.6.

We assume that x is the number of units produced and sold by the entire industry during a given time period and that $p = D(x) = -cx + d$, $c > 0$, is the price of one unit if x units are sold; that is, $p = -cx + d$ is the price of the x^{th} unit sold. We call $p = D(x)$ the **demand equation** and the graph the **demand curve**.

Estimating the demand equation is a fundamental problem for the management of any company or business. In the next example we consider the situation when just two data points are available and the demand equation is assumed to be linear.

EXAMPLE 4 **Finding the Demand Equation** Timmins estimated the municipal water demand in Delano, California. He estimated the demand x, measured in acre-feet (the volume of water needed to cover one acre of ground at a depth of one foot), with price p per acre-foot. He indicated two points on the demand curve, $(x, p) = (1500, 230)$ and $(x, p) = (5100, 50)$. Use this data to estimate the demand curve using a linear model. Estimate the price when the demand is 3000 acre-feet.

Source: Timmins 2002

Solution Figure 1.7 shows the two points $(x, p) = (1500, 230)$ and $(x, p) = (5100, 50)$ that lie on the demand curve. We are assuming that the demand curve is a straight line. The slope of the line is

$$m = \frac{50 - 230}{5100 - 1500} = -0.05$$

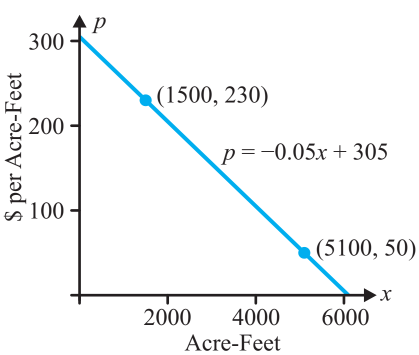


Figure 1.7

Now using the point-slope equation for a line with $(1500, 230)$ as the point on the line, we have

$$\begin{aligned} p - 230 &= m(x - 1500) \\ &= -0.05(x - 1500) \\ p &= -0.05x + 75 + 230 \\ &= -0.05x + 305 \end{aligned}$$

When demand is 3000 acre-feet, then $x = 3000$, and

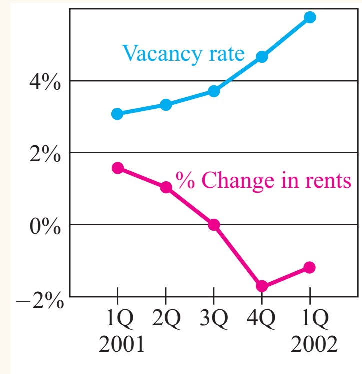
$$p = -0.05(3000) + 305 = 155$$

or \$155 per acre-foot. Thus, according to this model, if 3000 acre-feet is demanded, the price of each acre-foot will be \$155. ✦

CONNECTION
Demand for Apartments

The figure shows that during the minor recession of 2001, vacancy rates for apartments increased, that is, the demand for apartments decreased. Also notice from the figure that as demand for apartments decreased, rents also decreased. For example, in San Francisco's South Beach area, a two-bedroom apartment that had rented for $3000 a month two years before saw the rent drop to $2100 a month.

Source: Wall Street Journal, April 11, 2002



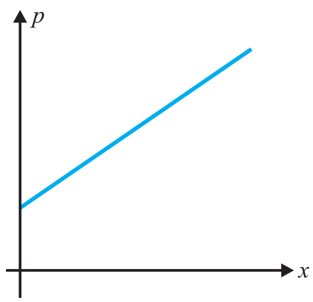


Figure 1.8
A supply equation

The **supply equation** $p = S(x)$ gives the price p necessary for suppliers to make available x units to the market. The graph of this equation is called the **supply curve**. A reasonable supply curve rises, moving from left to right, because the suppliers of any product naturally want to sell more if the price is higher. (See Shea 1993 who looked at a large number of industries and determined that the supply curve does indeed slope upward.) If the supply curve is linear, then as shown in Figure 1.8, the graph is a line sloping upward. Note the positive y-intercept. The y-intercept represents the **choke point** or lowest price a supplier is willing to accept.

EXAMPLE 5 **Finding the Supply Equation** Antle and Capalbo estimated a spring wheat supply curve. Use a mathematical model to determine a linear curve using their estimates that the supply of spring wheat will be 50 million bushels at a price of $2.90 per bushel and 100 million bushels at a price of $4.00 per bushel. Estimate the price when 80 million bushels is supplied.

Source: Antle and Capalbo 2001

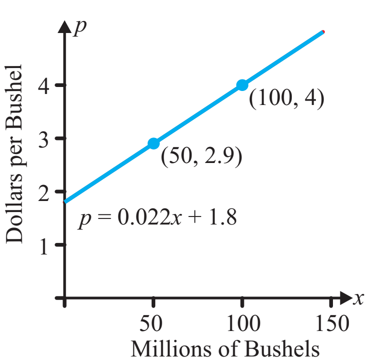


Figure 1.9

Solution Let x be in millions of bushels of wheat. We are then given two points on the linear supply curve, $(x, p) = (50, 2.9)$ and $(x, p) = (100, 4)$. The slope is

$$m = \frac{4 - 2.9}{100 - 50} = 0.022$$

The equation is then given by

$$p - 2.9 = 0.022(x - 50)$$

or $p = 0.022x + 1.8$. See Figure 1.9 and note that the line rises.

When supply is 80 million bushels, $x = 80$, and we have

$$p = 0.022(80) + 1.8 = 3.56$$

This gives a price of $3.56 per bushel. ✦

CONNECTION
Supply of Cotton

On May 2, 2002, the U.S. House of Representatives passed a farm bill that promises billions of dollars in subsidies to cotton farmers. With the prospect of a greater supply of cotton, cotton prices dropped 1.36 cents to 33.76 cents per pound.
Source: The Wall Street Journal, May 3, 2002.

✧ Straight-Line Depreciation

Many assets, such as machines or buildings, have a **finite** useful life and furthermore **depreciate** in value from year to year. For purposes of determining profits and taxes, various methods of depreciation can be used. In **straight-line depreciation** we assume that the value V of the asset is given by a **linear** equation in time t, say, $V = mt + b$. The slope m must be **negative** since the value of the asset **decreases** over time. The y-intercept is the initial value of the item and the slope gives the rate of depreciation (how much the item decreases in value per time period).

EXAMPLE 6 **Straight-Line Depreciation** A company has purchased a new grinding machine for $100,000 with a useful life of 10 years, after which it is assumed that the scrap value of the machine is $5000. Use straight-line depreciation to write an equation for the value V of the machine, where t is measured in years. What will be the value of the machine after the first year? After the second year? After the ninth year? What is the rate of depreciation?

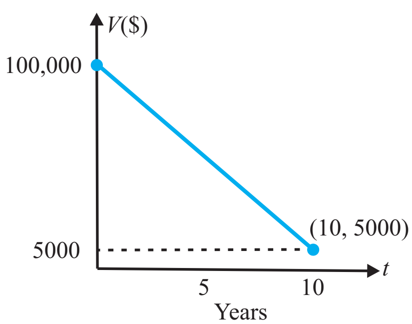


Figure 1.10

Solution We assume that $V = mt + b$, where m is the slope and b is the V-intercept. We then must find both m and b. We are told that the machine is initially worth $100,000, that is, when $t = 0$, $V = 100,000$. Thus, the point $(0,\ 100,000)$ is on the line, and 100,000 is the V-intercept, b. See Figure 1.10 and note the domain of t is $0 \leq t \leq 10$.

Since the value of the machine in 10 years will be $5000, this means that when $t = 10$, $V = 5000$. Thus, $(10, 5000)$ is also on the line. From Figure 1.10, the slope can then be calculated since we now know that the two points $(0,\ 100,000)$ and $(10, 5000)$ are on the line. Then

$$m = \frac{5000 - 100,000}{10 - 0} = -9500$$

Then, using the point-slope form of a line,

$$V = -9500t + 100,000$$

where the time t is in years since the machine was purchased and V is the value in dollars. Now we can find the value at different time periods,

$$V(1) = -9500(1) + 100,000 = 90,500 \text{ or } \$90,500$$
$$V(2) = -9500(2) + 100,000 = 81,000 \text{ or } \$81,000$$
$$V(9) = -9500(9) + 100,000 = 14,500 \text{ or } \$14,500$$

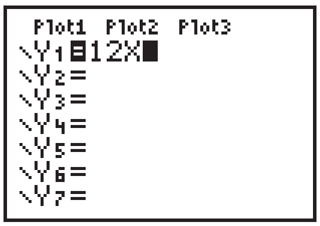
The rate of depreciation is the slope of the line, –$9500/year. ✦

Technology Option
Example 6 is solved using a graphing calculator in Technology Note 2 on page 12

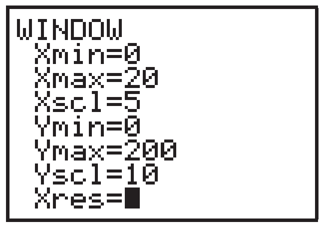
✧ Technology Corner

Technology Note 1 — Example 1 on a Graphing Calculator

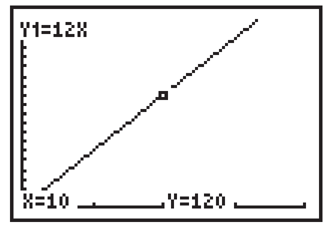
Begin by pressing the `Y=` button on the top row of your calculator. Enter 12 from the keypad and the variable X using the `X,T,θ,n` button. The result is shown in Screen 1.1. Next choose the viewing window by pressing the `WINDOW` button along the top row of buttons. The smallest value for x is 0 (no steak specials sold), so enter 0 for `Xmin`. To evaluate the function for 12 steak specials, or $x = 12$, choose an `Xmax` that is greater than 12. We have chosen `Xmax=20` and `Xscl=5` (to have a tick mark is placed every 5 units on the `X`-axis). The range of values for y must be large enough to view the function. The `Y` range was set as `Ymin=0`, `Ymax=200` and `Yscl=10`. The `Xres` setting can be left at 1 to have the full resolution on the screen. Screen 1.2 shows the window settings.



Screen 1.1



Screen 1.2

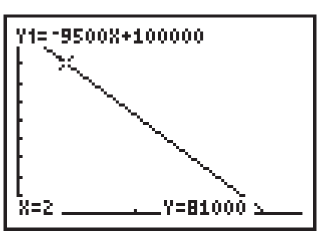


Screen 1.3
[0, 20]×[0, 200]

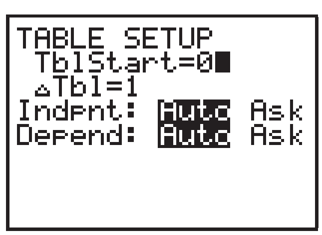
In later graphs, the window will be listed under the graph as `[Xmin, Xmax]`×`[Ymin, Ymax]`. The choice of the `Xscl` and `Yscl` will be left to the reader. Press the `GRAPH` button to see the function displayed. To find the value of our function at a particular x-value, choose the `CALC` menu (above the `TRACE` button). Avoid the trace function as it will not go to an exact x-value. Choose the first option, `1:value` and then enter the value 10. Pressing enter again to evaluate, we see in Screen 1.3 the value of the function at $x = 10$ is 120.

Technology Note 2 — Example 6 on a Graphing Calculator

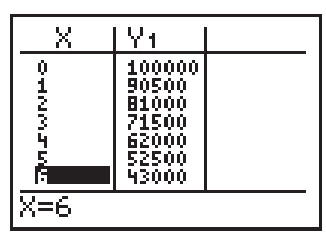
The depreciation function can be graphed as done in Technology Note 1 above. Screen 1.4 shows the result of graphing Y_1`=-9500X+100000` and finding the value at `X=2`. We were asked to find the value of the grinding machine at several different times; the table function can be used to simplify this task. Once a function is entered, go to the `TBLSET` feature by pressing `2ND` and then `WINDOW` (see Screen 1.5). We want to start at `X=0` and count by 1's, so set `TblStart = 0` and `ΔTbl=1`. To see the table, press `2ND` and then `GRAPH` (see Screen 1.6).



Screen 1.4
[0, 12]×[0, 100000]



Screen 1.5



Screen 1.6

REMARK: You can also find a window by entering `Xmin=0` and `Xmax=10`, the known domain of this function, and then pressing `ZOOM` and scrolling to choose `0:ZoomFit`. This useful feature will evaluate the functions to be graphed from `Xmin` to `Xmax` and choose the values for `Ymin` and `Ymax` to allow the functions to be seen.

Self-Help Exercises 1.1

1. Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical medium-sized plant they estimated fixed costs at \$400,000 and estimated the cost of each ton of fertilizer was \$200 to produce. The plant sells its fertilizer output at \$250 per ton.

a. Find and graph the cost, revenue, and profit equations.

b. Determine the cost, revenue, and profits when the number of tons produced and sold is 5000, 7000, and 9000 tons.

Source: Rogers and Akridge 1996

2. The excess supply and demand curves for wheat worldwide were estimated by Schmitz and coworkers to be

Supply: $p = 7x - 400$

Demand: $p = 510 - 3.5x$

where p is price in dollars per metric ton and x is in millions of metric tons. Excess demand refers to the excess of wheat that producer countries have over their own consumption. Graph these two functions. Find the prices for the supply and demand models when x is 70 million metric tons. Is the price for supply or demand larger? Repeat these questions when x is 100 million metric tons.

Source: Schmitz, Sigurdson, and Doering 1986

1.1 Exercises

In Exercises 1 and 2 you are given the cost per item and the fixed costs. Assuming a linear cost model, find the cost equation, where C is cost and x is the number produced.

1. Cost per item = \$3, fixed cost=\$10,000

2. Cost per item = \$6, fixed cost=\$14,000

In Exercises 3 and 4 you are given the price of each item, which is assumed to be constant. Find the revenue equation, where R is revenue and x is the number sold.

3. Price per item = \$5

4. Price per item = \$0.10

5. Using the cost equation found in Exercise 1 and the revenue equation found in Exercise 3, find the profit equation for P, assuming that the number produced equals the number sold.

6. Using the cost equation found in Exercise 2 and the revenue equation found in Exercise 4, find the profit equation for P, assuming that the number produced equals the number sold.

In Exercises 7 to 10, find the demand equation using the given information.

7. A company finds it can sell 10 items at a price of \$8 each and sell 15 items at a price of \$6 each.

8. A company finds it can sell 40 items at a price of \$60 each and sell 60 items at a price of \$50 each.

9. A company finds that at a price of \$35, a total of 100 items will be sold. If the price is lowered by \$5, then 20 additional items will be sold.

10. A company finds that at a price of \$200, a total of 30 items will be sold. If the price is raised \$50, then 10 fewer items will be sold.

In Exercises 11 to 14, find the supply equation using the given information.

11. A supplier will supply 50 items to the market if the price is \$95 per item and supply 100 items if the price is \$175 per item.

12. A supplier will supply 1000 items to the market if the price is \$3 per item and supply 2000 items if the price is \$4 per item.

13. At a price of \$60 per item, a supplier will supply 10 of these items. If the price increases by \$20, then 4 additional items will be supplied.

14. At a price of \$800 per item, a supplier will supply 90 items. If the price decreases by \$50, then the supplier will supply 20 fewer items.

In Exercises 15 to 18, find the depreciation equation and corresponding domain using the given information.

15. A calculator is purchased for \$130 and the value decreases by \$15 per year for 7 years.

16. A violin bow is purchased for \$50 and the value decreases by \$5 per year for 6 years.

17. A car is purchased for \$15,000 and is sold for \$6000 six years later.

18. A car is purchased for \$32,000 and is sold for \$23,200 eight years later.

Applications

19. Wood Chipper Cost A contractor needs to rent a wood chipper for a day for \$150 plus \$10 per hour. Find the cost function.

20. Truck Rental Cost A builder needs to rent a dump truck for a day for \$75 plus \$0.40 per mile. Find the cost function.

21. Sewing Machine Cost A shirt manufacturer is considering purchasing a sewing machine for \$91,000 and it will cost \$2 to sew each of their standard shirts. Find the cost function.

22. Copying Cost At Lincoln Library there are two ways to pay for copying. You can pay 5 cents a copy, or you can buy a plastic card for \$5 and then pay 3 cents a copy. Let x be the number of copies you make. Write an equation for your costs for each way of paying.

23. Assume that the linear cost model applies and fixed costs are \$1000. If the total cost of producing 800 items is \$5000, find the cost equation.

24. Assume that the linear cost model applies. If the total cost of producing 1000 items at \$3 each is \$5000, find the cost equation.

25. When 50 silver beads are ordered they cost \$1.25 each. If 100 silver beads are ordered, they cost \$1.00 each. How much will each silver bead cost if 250 are ordered?

26. You find that when you order 75 magnets, the average cost per magnet is \$0.90 and when you order 200 magnets, the average cost per magnet is \$0.80. What is the cost equation for these custom magnets?

27. Assume that the linear revenue model applies. If the total revenue from selling 600 items is \$7200, find the revenue equation.

28. Assume that the linear revenue model applies. If the total revenue from selling 1000 items is \$8000, find the revenue equation.

29. Assume that the linear cost and revenue model applies. An item sells for \$10. If fixed costs are \$2000 and profits are \$7000 when 1000 items are made and sold, find the cost equation.

30. Assume that the linear cost and revenue models applies. An item that costs \$3 to make sells for \$6. If profits of \$5000 are made when 2000 items are made and sold, find the cost equation.

31. Assume that the linear cost and revenue models applies. An item costs \$3 to make. If fixed costs are \$1000 and profits are \$7000 when 1000 items are made and sold, find the revenue equation.

32. Assume that the linear cost and revenue models applies. An item costs \$7 to make. If fixed costs are \$1500 and profits are \$1700 when 200 items are made and sold, find the revenue equation.

33. Demand for Blueberries A grocery store sells 27 packages of blueberries daily when the price is \$3.18 per package. If the price is decreased by \$0.25 per package, then the store will sell an additional 5 packages every day. What is the demand equation for blueberries?

34. Demand for Bagels A bakery sells 124 bagels daily when the price is \$1.50 per bagel. If the price is increased by \$0.50, the bakery will sell 25 fewer bagels. What is the demand equation for bagels?

35. **Supply of Basil** A farmer is willing to supply 15 packages of organic basil to a market for \$2 per package. If the market offers the farmer \$1 more per package, the farmer will supply 20 more packages of organic basil. What is the supply equation for organic basil?

36. **Supply of Roses** A grower is willing to supply 200 long-stemmed roses per week to a florist for \$0.85 per rose. If the florist offers the grower \$0.20 less per rose, then the grower will supply 50 fewer roses. What is the demand equation for these long-stemmed roses?

37. **Machine Depreciation** Consider a new machine that costs \$50,000 and has a useful life of nine years and a scrap value of \$5000. Using straight-line depreciation, find the equation for the value V in terms of t, where t is in years. Find the value after one year and after five years.

38. **Building Depreciation** A new building that costs \$1,100,000 has a useful life of 50 years and a scrap value of \$100,000. Using straight-line depreciation, find the equation for the value V in terms of t, where t is in years. Find the value after 1 year, after 2 years, and after 40 years.

Referenced Applications

39. **Cotton Ginning Cost** Misra and colleagues estimated the cost function for the ginning industry in the Southern High Plains of Texas. They give a (total) cost function C by $C(x) = 21x + 674,000$, where C is in dollars and x is the number of bales of cotton. Find the fixed and variable costs.

Source: Misra, McPeek, and Segarra 2000

40. **Fishery Cost** The cost function for wild crayfish was estimated by Bell to be a function $C(x)$, where x is the number of millions of pounds of crayfish caught and C is the cost in millions of dollars. Two points that are on the graph are $(x, C) = (8, 0.157)$ and $(x, C) = (10, 0.190)$. Using this information and assuming a linear model, determine a cost function.

Source: Bell 1986

41. **Costs of Manufacturing Fenders** Saur and colleagues did a careful study of the cost of manufacturing automobile fenders using five different materials: steel, aluminum, and three injection-molded polymer blends: rubber-modified polypropylene (RMP), nylon-polyphenylene oxide (NPN), and polycarbonate-polybutylene terephthalate (PPT). The following table gives the fixed and variable costs of manufacturing each pair of fenders.

Costs	Steel	Aluminum	RMP	NPN
Variable	\$5.26	\$12.67	\$13.19	\$9.53
Fixed	\$260,000	\$385,000	\$95,000	\$95,000

Write down the cost function associated with each of the materials.

Source: Saur, Fava, and Spatari 2000

42. **Cost of Raising a Steer** Kaitibie and colleagues estimated the costs of raising a young steer purchased for \$428 and the variable food cost per day for \$0.67. Determine the cost function based on the number of days this steer is grown.

Source: Kaitibie, Epplin, Brorsen, Horn, Eugene G. Krenzer, and Paisley 2003

43. **Revenue for red wine grapes in Napa Valley** Brown and colleagues report that the price of red varieties of grapes in Napa Valley was \$2274 per ton. Determine a revenue function and indicate the independent and dependent variables.

Source: Brown, Lynch, and Zilberman 2002

44. **Revenue for wine grapes in Napa Valley** Brown and colleagues report that the price of certain wine grapes in Napa Valley was \$617 per ton. They estimated that 6 tons per acre was yielded. Determine a revenue function using the independent variable as the number of acres.

Source: Brown, Lynch, and Zilberman 2002

45. **Ecotourism Revenue** Velazquez and colleagues studied the economics of ecotourism. A grant of \$100,000 was given to a certain locality to use to develop an ecotourism alternative to destroying forest and the consequent biodiversity. The community found that each visitor spent \$40 on average. If x is the number of visitors, find a revenue function. How many visitors are needed to reach the initial \$100,000 invested? (This community was experiencing about 2500 visits per year.)

Source: Velazquez, Bocco, and Torres 2001

46. Heinz Ketchup Revenue Besanko and colleagues reported that a Heinz ketchup 32-oz size yielded a price of \$0.043 per ounce. Write an equation for revenue as a function of the number of 32-oz bottles of Heinz ketchup.

Source: Besanko, Dubé, and Gupta 2003

47. Fishery Revenue Grafton created a mathematical model for revenue for the northern cod fishery. We can see from this model that when 150,000 kilograms of cod were caught, \$105,600 of revenue were yielded. Using this information and assuming a linear revenue model, find a revenue function R in units of \$1000 where x is given in units of 1000 kilograms.

Source: Grafton, Sandal, and Steinhamn 2000

48. Shrimp Profit Kekhora and McCann estimated a cost function for a shrimp production function in Thailand. They gave the fixed costs per hectare of \$1838 and the variable costs per hectare of \$14,183. The revenue per hectare was given as \$26,022

a. Determine the total cost for 1 hectare.

b. Determine the profit for 1 hectare.

Source: Kekhora and McCann 2003

49. Rice Production Profit Kekhora and McCann estimated a cost function for the rice production function in Thailand. They gave the fixed costs per hectare of \$75 and the variable costs per hectare of \$371. The revenue per hectare was given as \$573.

a. Determine the total cost for 1 hectare.

b. Determine the profit for 1 hectare.

Source: Kekhora and McCann 2003

50. Profit for Small Fertilizer Plants In 1996 Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical small-sized plant they estimated fixed costs at \$235,487 and estimated that it cost \$206.68 to produce each ton of fertilizer. The plant sells its fertilizer output at \$266.67 per ton. Find the cost, revenue, and profit equations.

Source: Rogers and Akridge 1996

51. Profit for Large Fertilizer Plants In 1996 Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical large-sized plant they estimated fixed costs at \$447,917 and estimated that it cost \$209.03 to produce each ton of fertilizer. The plant sells its fertilizer output at \$266.67 per ton. Find the cost, revenue, and profit equations.

Source: Rogers and Akridge 1996

52. Demand for Recreation Shafer and others estimated a demand curve for recreational power boating in a number of bodies of water in Pennsylvania. They estimated the price p of a power boat trip including rental cost of boat, cost of fuel, and rental cost of equipment. For the Lake Erie/Presque Isle Bay Area they collected data indicating that for a price (cost) of \$144, individuals made 10 trips, and for a price of \$50, individuals made 20 trips. Assuming a linear model determine the demand curve. For 15 trips, what was the cost?

Source: Shafer, Upneja, Seo, and Yoon 2000

53. Demand for Recreation Shafer and others estimated a demand curve for recreational power boating in a number of bodies of water in Pennsylvania. They estimated the price p of a power boat trip including rental cost of boat, cost of fuel, and rental cost of equipment. For the Three Rivers Area they collected data indicating that for a price (cost) of \$99, individuals made 10 trips, and for a price of \$43, individuals made 20 trips. Assuming a linear model determine the demand curve. For 15 trips, what was the cost?

Source: Shafer, Upneja, Seo, and Yoon 2000

54. Demand for Cod Grafton created a mathematical model for demand for the northern cod fishery. We can see from this model that when 100,000 kilograms of cod were caught the price was \$0.81 per kilogram and when 200,000 kilograms of cod were caught the price was \$0.63 per kilogram. Using this information and assuming a linear demand model, find a demand function.

Source: Grafton, Sandal, and Steinhamn 2000

55. Demand for Rice Suzuki and Kaiser estimated the demand equation for rice in Japan to be $p = 1,195,789 - 0.1084753x$, where x is in tons of rice and p is in yen per ton. In 1995, the quantity of rice consumed in Japan was 8,258,000 tons.

a. According to the demand equation, what was the price in yen per ton?

b. What happens to the price of a ton of rice when the demand increases by 1 ton. What has this number to do with the demand equation?

Source: Suzuki and Kaiser 1998

56. Supply of Childcare Blau and Mocan gathered data over a number of states and estimated a supply curve that related quality of child care with price. For quality q of child care they developed an index of quality and for price p they used their own units. In their graph they gave $q = S(p)$, that is, the price was the independent variable. On this graph we see the following points: $(p, q) = (1, 2.6)$ and $(p, q) = (3, 5.5)$. Use this information and assuming a linear model, determine the supply curve.

Source: Blau and Mocan 2002

57. Oil Production Technology D'Unger and coworkers studied the economics of conversion to saltwater injection for inactive wells in Texas. (By injecting saltwater into the wells, pressure is applied to the oil field, and oil and gas are forced out to be recovered.) The expense of a typical well conversion was estimated to be \$31,750. The monthly revenue as a result of the conversion was estimated to be \$2700. If x is the number of months the well operates after conversion, determine a revenue function as a function of x. How many months of operation would it take to recover the initial cost of conversion?

Source: D'Unger, Chapman, and Carr 1996

58. Rail Freight In a report of the Federal Trade Commission (FTC) an example is given in which the Portland, Oregon, mill price of 50,000 board square feet of plywood is \$3525 and the rail freight is \$0.3056 per mile.

a. If a customer is located x rail miles from this mill, write an equation that gives the total freight f charged to this customer in terms of x for delivery of 50,000 board square feet of plywood.

b. Write a (linear) equation that gives the total c charged to a customer x rail miles from the mill for delivery of 50,000 board square feet of plywood. Graph this equation.

c. In the FTC report, a delivery of 50,000 board square feet of plywood from this mill is made to New Orleans, Louisiana, 2500 miles from the mill. What is the total charge?

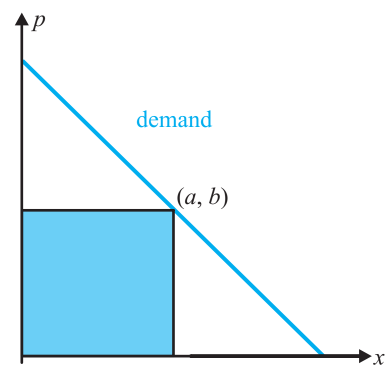
Source: Gilligan 1992

Extensions

59. Understanding the Revenue Equation Assuming a linear revenue model, explain in a complete sentence where you expect the y-intercept to be. Give a reason for your answer.

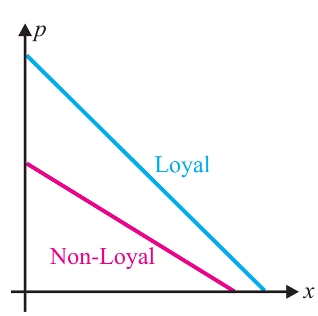
60. Understanding the Cost and Profit Equations Assuming a linear cost and revenue model, explain in complete sentences where you expect the y-intercepts to be for the cost and profit equations. Give reasons for your answers.

61. Demand Area In the figure we see a demand curve with a point (a, b) on it. We also see a rectangle with a corner on this point. What do you think the area of this rectangle represents?



62. Demand Curves for Customers Price and Connor studied the difference between demand curves between loyal customers and nonloyal customers in ready-to-eat cereal. The figure shows two such as demand curves. (Note that the independent variable is the quantity.) Discuss the differences and the possible reasons. For example, why do you think that the p-intercept for the loyal demand curve is higher than the other? Why do you think the loyal demand is above the other? What do you think the producers should do to make their customers more loyal?

Source: Price and Conner 2003



63. Cost of Irrigation Water Using an argument that is too complex to give here, Tolley and Hastings argued that if c is the cost in 1960 dollars per acre-foot of water in the area of Nebraska and x is the acre-feet of water available, then $c = 12$ when $x = 0$. They also noted that farms used about 2 acre-feet of water in the Ainsworth area when this water was free. If we assume (as they did) that the relationship between c and x is linear, then find the equation that c and x must satisfy.

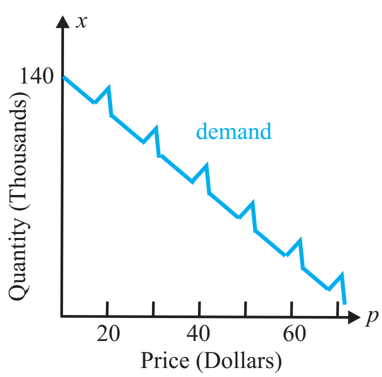
Source: Tolley and Hastings 1960

64. Kinked and Spiked Demand and Profit Curves Stiving determined demand curves. Note that a manufacturer can decide to produce a durable good with a varying quality.

a. The figure shows a demand curve for which the quality of an item depends on the price. Explain if this demand curve seems reasonable.

b. Notice that the demand curve is kinked and spiked at prices at which the price ends in the digit, such at \$39.99. Explain why you think this could happen.

Source: Stiving 2000



65. Profits for Kansas Beef Cow Farms Featherstone and coauthors studied 195 Kansas beef cow farms. The average fixed and variable costs are found in the following table.

Variable and Fixed Costs	
Costs per cow	
Feed costs	\$261
Labor costs	\$82
Utilities and fuel costs	\$19
Veterinary expenses costs	\$13
Miscellaneous costs	\$18
Total variable costs	\$393
Total fixed costs	\$13,386

The farm can sell each cow for \$470. Find the cost, revenue, and profit functions for an average farm. The average farm had 97 cows. What was the profit for 97 cows? Can you give a possible explanation for your answer?

Source: Featherstone, Langemeier, and Ismet 1997

66. Profit on Corn Roberts formulated a mathematical model of corn yield response to nitrogen fertilizer in high-yield response land given by $Y(N)$, where Y is bushels of corn per acre and N is pounds of nitrogen per acre. They estimated that the farmer obtains \$2.42 for a bushel of corn and pays \$0.22 a pound for nitrogen fertilizer. For this model they assume that the only cost to the farmer is the cost of nitrogen fertilizer.

a. We are given that $Y(20) = 47.8$ and $Y(120) = 125.8$. Find $Y(N)$.

b. Find the revenue $R(N)$.

c. Find the cost $C(N)$.

d. Find the profit $P(N)$.

Source: Roberts, English, and Mahajashetti 2000

67. Profit in the Cereal Manufacturing Industry Cotterill estimated the costs and prices in the cereal-manufacturing industry. The table summarizes the costs in both pounds and tons in the manufacture of a typical cereal.

Item	\$/lb	\$/ton
Manufacturing cost:		
Grain	0.16	320
Other ingredients	0.20	400
Packaging	0.28	560
Labor	0.15	300
Plant costs	0.23	460
Total manufacturing costs	1.02	2040
Marketing expenses:		
Advertising	0.31	620
Consumer promo (mfr. coupons)	0.35	700
Trade promo (retail in-store)	0.24	480
Total marketing costs	0.90	1800
Total variable costs	1.92	3840

The manufacturer obtained a price of \$2.40 a pound, or \$4800 a ton. Let x be the number of tons of cereal manufactured and sold and let p be the price of a ton sold. Nero estimated fixed costs for a typical plant to be \$300 million. Let the cost, revenue, and profits be given in thousands of dollars. Find the cost, revenue, and profit equations. Also make a table of values for cost, revenue, and profit for production levels of 200,000, 300,000 and 400,000 tons and discuss what the table of numbers is telling you.

Source: Cotterill and Haller 1997 and Nero 2001

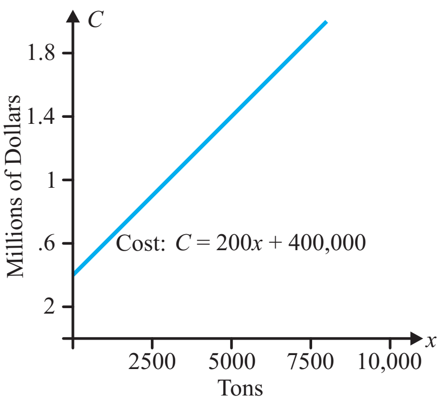
Solutions to Self-Help Exercises 1.1

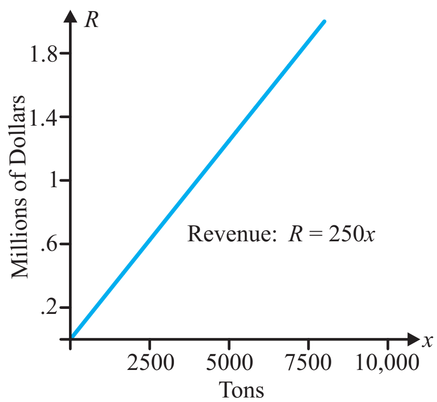
1. Let x be the number of tons of fertilizer produced and sold.

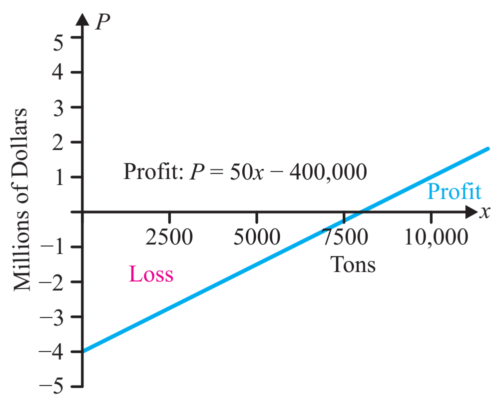
a. Then the cost, revenue and profit equations are

$$\begin{aligned} C(x) &= (\text{variable cost}) + (\text{fixed cost}) \\ &= 200x + 400,000 \\ R(x) &= (\text{price per ton}) \times (\text{number of tons sold}) \\ &= 250x \\ P(x) &= R - C \\ &= (250x) - (200x + 400,000) \\ &= 50x - 400,000 \end{aligned}$$

The cost, revenue, and profit equations are graphed below.







b. If $x = 5000$, then

$$\begin{aligned} C(5000) &= 200(5000) + 400,000 = 1,400,000 \\ R(5000) &= 250(5000) = 1,250,000 \\ P(5000) &= 1,250,000 - 1,400,000 = -150,000 \end{aligned}$$

If 5000 tons are produced and sold, the cost is $1,400,000, the revenue is $1,250,000, and there is a *loss* of $150,000. Doing the same for some other values of x, we have the following,

x	5000	7000	9000
Cost	1,400,000	1,800,000	2,200,000
Revenue	1,250,000	1,750,000	2,250,000
Profit (or loss)	−150,000	−50,864	50,130

2. The graphs are shown in the figure. When $x = 70$, we have

supply: $p = 7(70) - 400 = 49$

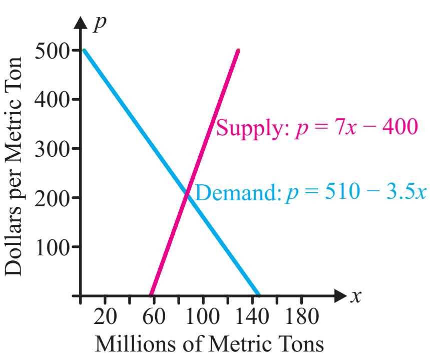
demand: $p = 510 - 3.5(70) = 265$

Demand is larger. When $x = 100$, we have

supply: $p = 7(100) - 400 = 300$

demand: $p = 510 - 3.5(100) = 160$

Supply is larger.



1.2 Systems of Linear Equations

APPLICATION
Cost, Revenue, and Profit Models

In Example 3 in the last section we found the cost and revenue equations in the dress-manufacturing industry. Let x be the number of dresses made and sold. Recall that cost and revenue functions were found to be $C(x) = 25x + 80,000$ and $R(x) = 75x$. Find the point at which the profit is zero. See Example 2 for the answer.

We now begin to look at systems of linear equations in many unknowns. In this section we first consider systems of two linear equations in two unknowns. We will see that solutions of such a system have a variety of applications.

✧ Two Linear Equations in Two Unknowns

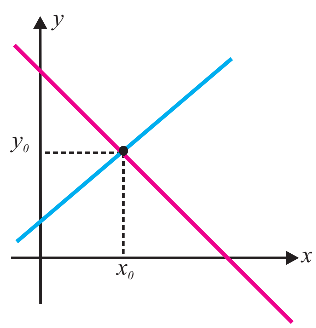


Figure 1.11

In this section we will encounter applications that have a unique solution to a system of two linear equations in two unknowns. For example, consider two lines,

$$L_1 : y = m_1x + b_1$$
$$L_2 : y = m_2x + b_2$$

If these two linear equations are not parallel ($m_1 \neq m_2$), then the lines must intersect at a unique point, say (x_0, y_0) as shown in Figure 1.11. This means that (x_0, y_0) is a **solution** to the two linear equations and must satisfy both of the equations

$$y_0 = m_1x_0 + b_1$$
$$y_0 = m_2x_0 + b_2$$

EXAMPLE 1 **Intersection of Two Lines** Find the solution (intersection) of the two lines.

$$L_1 : y = 7x - 3$$
$$L_2 : y = -4x + 9$$

Solution To find the solution, set the two lines equal to each other, $L_1 = L_2$,

$$y_0 = y_0$$
$$7x_0 - 3 = 4x_0 + 9$$
$$11x_0 = 12$$
$$x_0 = \frac{12}{11}$$

To find the value of y_0, substitute the x_0 value into either equation,

$$y_0 = 7\left(\frac{12}{11}\right) - 3 = \frac{51}{11}$$
$$y_0 = 4\left(\frac{12}{11}\right) + 9 = \frac{51}{11}$$

So, the solution to this system is the intersection point $(12/11, 51/11)$. ✦

Technology Option
Example 1 is solved using a graphing calculator in Technology Note 1 on page 25

✧ Decision Analysis

In the last section we considered linear mathematical models of cost, revenue, and profit for a firm. In Figure 1.12 we see the graphs of two typical cost and revenue functions. We can see in this figure that for smaller values of x, the cost line is **above** the revenue line and therefore the profit P is **negative**. Thus the firm has losses. As x becomes larger, the revenue line becomes **above** the cost line and therefore the profit becomes **positive**. The value of x at which the profit is zero is called the **break-even quantity**. Geometrically, this is the point of intersection of the cost line and the revenue line. Mathematically, this requires us to solve the equations $y = C(x)$ and $y = R(x)$ simultaneously.

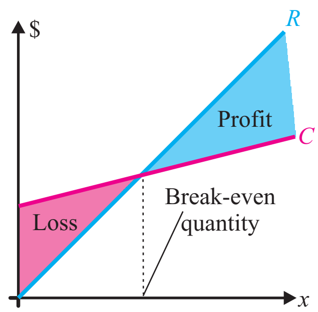


Figure 1.12

EXAMPLE 2 Finding the Break-Even Quantity In Example 3 in the last section we found the cost and revenue equations in a dress-manufacturing firm. Let x be the number of dresses manufactured and sold and let the cost and revenue be given in dollars. Then recall that the cost and revenue equations were found to be $C(x) = 25x + 80,000$ and $R(x) = 75x$. Find the break-even quantity.

Technology Option
Example 2 is solved using a graphing calculator in Technology Note 2 on page 26

Solution To find the break-even quantity, we need to solve the equations $y = C(x)$ and $y = R(x)$ simultaneously. To do this we set $R(x) = C(x)$. Doing this we have

$$\begin{aligned} R(x) &= C(x) \\ 75x &= 25x + 80,000 \\ 50x &= 80,000 \\ x &= 1600 \end{aligned}$$

Thus, the firm needs to produce and sell 1600 dresses to break even (i.e., for profits to be zero). See Figure 1.13. ✦

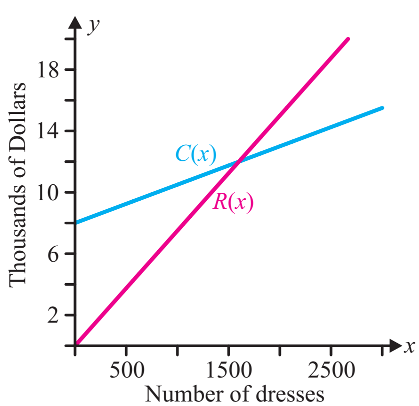


Figure 1.13

REMARK: Notice that $R(1600) = 120,000 = C(1600)$ so it costs the company \$120,000 to make the dresses and they bring in \$120,000 in revenue when the dresses are all sold.

In the following example we consider the total energy consumed by automobile fenders using two different materials. We need to decide how many miles carrying the fenders result in the same energy consumption and which type of fender will consume the least amount of energy for large numbers of miles.

EXAMPLE 3 **Break-Even Analysis** Saur and colleagues did a careful study of the amount of energy consumed by each type of automobile fender using various materials. The total energy was the sum of the energy needed for production plus the energy consumed by the vehicle used in carrying the fenders. If x is the miles traveled, then the total energy consumption equations for steel and rubber-modified polypropylene (RMP) were as follows:

$$\text{Steel: } E = 225 + 0.012x$$
$$\text{RPM: } E = 285 + 0.007x$$

Graph these equations, and find the number of miles for which the total energy consumed is the same for both fenders. Which material uses the least energy for 15,000 miles?

Source: Saur, Fava, and Spatari 2000

Solution The total energy using steel is $E_1(x) = 225+0.012x$ and for RMP is $E_2(x) = 285 + 0.007x$. The graphs of these two linear energy functions are shown in Figure 1.14. We note that the graphs intersect. To find this intersection we set $E_1(x) = E_2(x)$ and obtain

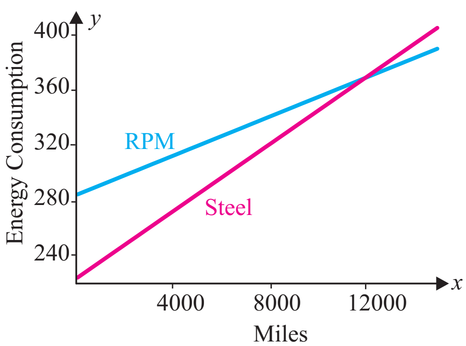


Figure 1.14

$$\begin{aligned} E_1(x) &= E_2(x) \\ 225 + 0.012x &= 285 + 0.007x \\ 0.005x &= 60 \\ x &= 12,000 \end{aligned}$$

So, 12,000 miles results in the total energy used by both materials being the same.

Setting $x = 0$ gives the energy used in production, and we note that steel uses less energy to produce these fenders than does RPM. However, since steel is heavier than RPM, we suspect that carrying steel fenders might require more total energy when the number of pair of fenders is large. Indeed, we see in Figure 1.14 that the graph corresponding to steel is above that of RPM when $x > 12,000$. Checking this for $x = 15,000$, we have

$$\begin{aligned} \text{steel:} \qquad E_1(x) &= 225 + 0.012x \\ E_1(15,000) &= 225 + 0.012(15,000) \\ &= 405 \\ \text{RPM:} \qquad E_2(x) &= 285 + 0.007x \\ E_2(15,000) &= 285 + 0.007(15,000) \\ &= 390 \end{aligned}$$

So for traveling 15,000 miles, the total energy used by RPM is less than that for steel. ✦

✧ Supply and Demand Equilibrium

The best-known law of economics is the law of supply and demand. Figure 1.15 shows a demand equation and a supply equation that intersect.

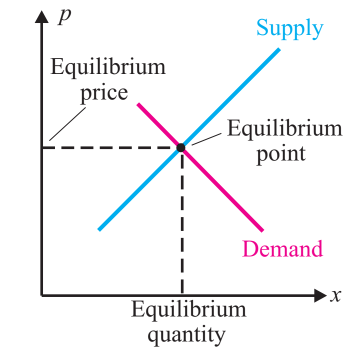


Figure 1.15

The point of intersection, or the point at which supply equals demand, is called the **equilibrium point**. The x-coordinate of the equilibrium point is called the **equilibrium quantity**, x_0, and the p-coordinate is called the **equilibrium price**, p_0. In other words, at a price p_0, the consumer is willing to buy x_0 items and the producer is willing to supply x_0 items.

EXAMPLE 4 Finding the Equilibrium Point Tauer determined demand and supply curves for milk in this country. If x is billions of pounds of milk and p is in dollars per hundred pounds, he found that the demand function for milk was $p = D(x) = 56 - 0.3x$ and the supply function was $p = S(x) = 0.1x$. Graph the demand and supply equations. Find the equilibrium point.

Source: Tauer 1994

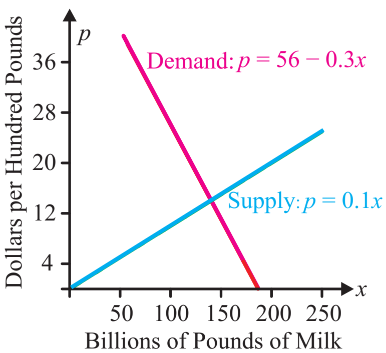


Figure 1.16

Solution The demand equation $p = D(x) = 56 - 0.3x$ is a line with negative slope -0.3 and y-intercept 56 and is graphed in Figure 1.16. The supply equation $p = S(x) = 0.1x$ is a line with positive slope 0.1 with y-intercept 0. This is also graphed in Figure 1.16.

To find the point of intersection of the demand curve and the supply curve, set $S(x) = D(x)$ and solve:

$$\begin{aligned} S(x) &= D(x) \\ 0.1x &= 56 - 0.3x \\ 0.4x &= 56 \\ x &= 140 \end{aligned}$$

Then since $p(x) = 0.1x$,

$$p(140) = 0.1(140) = 14$$

We then see that the equilibrium point is $(x, p) = (140, 14)$. That is, 140 billions pounds of milk at \$14 per hundred pounds of milk. ✦

EXAMPLE 5 Supply and Demand Refer to Example 4. What will consumers and suppliers do if the price is $p_1 = 25$ shown in Figure 1.17a? What if the price is $p_2 = 5$ as shown in Figure 1.17b?

Solution If the price is at $p_1 = 25$ shown in Figure 1.17a, then let the supply of milk be denoted by x_S. Let us find x_S.

$$\begin{aligned} p &= S(x_S) \\ 25 &= 0.1x_S \\ x_S &= 250 \end{aligned}$$

That is, 250 billion pounds of milk will be supplied. Keeping the same price of $p_1 = 25$ shown in Figure 1.17a, then let the demand of milk be denoted by x_D. Let us find x_D. Then

$$\begin{aligned} p &= D(x_D) \\ 25 &= 56 - 0.3x_D \\ x_D &\approx 103 \end{aligned}$$

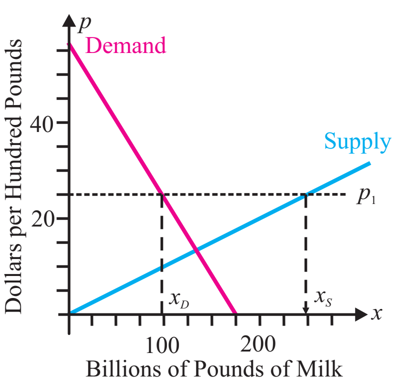


Figure 1.17a

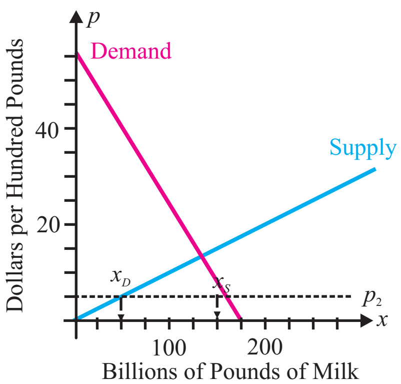


Figure 1.17b

So only 103 billions of pounds of milk are demanded by consumers. There will be a surplus of $250 - 103 = 147$ billions of pounds of milk. To work off the surplus, the price should fall toward the equilibrium price of $p_0 = 14$.

If the price is at $p_2 = 5$ shown in Figure 1.17b, then let the supply of milk be denoted by x_S. Let us find x_S. Then

$$\begin{aligned} p &= S(x_S) \\ 5 &= 0.1x_S \\ x_S &= 50 \end{aligned}$$

That is, 50 billion pounds of milk will be supplied. Keeping the same price of $p_2 = 5$ shown in Figure 1.17b, then let the demand of milk be denoted by x_D. Let us find x_D. Then

$$\begin{aligned} p &= D(x_D) \\ 5 &= 56 - 0.3x_D \\ x_D &\approx 170 \end{aligned}$$

So 170 billions of pounds of milk are demanded by consumers. There will be a shortage of $170 - 50 = 120$ billions of pounds of milk, and the price should rise toward the equilibrium price. ✦

CONNECTION
Demand for Steel Outpaces Supply

In early 2002 President George W. Bush imposed steep tariffs on imported steel to protect domestic steel producers. As a result millions of tons of imported steel were locked out of the country. Domestic steelmakers announced on March 27, 2002, that they had been forced to ration steel to their customers and boost prices because demand has outpaced supply.

Source: Wall Street Journal

✧ Enrichment: Decision Analysis Complications

In the following example we look at the cost of manufacturing automobile fenders using two different materials. We determine the number of pairs of fenders that will be produced by using the same cost. However, we must keep in mind that we do not produce **fractional** numbers of fenders, but rather only **whole** numbers. For example, we can produce one or two pairs of fenders, but not 1.43 pairs.

EXAMPLE 6 **Decision Analysis for Manufacturing Fenders** Saur and colleagues did a careful study of the cost of manufacturing automobile fenders using two different materials: steel and a rubber-modified polypropylene blend (RMP). The following table gives the fixed and variable costs of manufacturing each pair of fenders.

Variable and Fixed Costs of Pairs of Fenders

Costs	Steel	RMP
Variable	\$5.26	\$13.19
Fixed	\$260,000	\$95,000

Graph the cost function for each material. Find the number of fenders for which the cost of each material is the same. Which material will result in the lowest cost if a large number of fenders are manufactured?

Source: Saur, Fava, and Spatari 2000

Solution The cost function for steel is $C_1(x) = 5.26x + 260,000$ and for RMP is $C_2(x) = 13.19x + 95,000$. The graphs of these two cost functions are shown in Figure 1.18. For a small number of fenders, we see from the graph that the cost for steel is greater than that for RMP. However, for a large number of fenders the cost for steel is less. To find the number of pairs that yield the same cost for each material, we need to solve $C_2(x) = C_1(x)$.

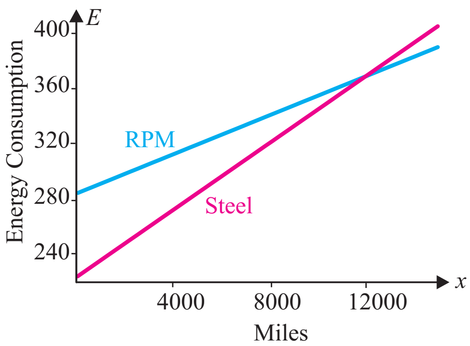


Figure 1.18

$$\begin{aligned} C_2(x) &= C_1(x) \\ 13.19x + 95,000 &= 5.26x + 260,000 \\ 7.93x &= 165,000 \\ x &= 20,807.062 \end{aligned}$$

This is a real application, so only an **integer** number of fenders can be manufactured. We need to round off the answer given above and obtain 20,807 pair of fenders. ✦

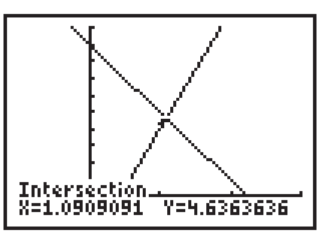
REMARK: Note that $C_2(20,807) = 369,444.44$ and $C_1(20,807) = 369,444.82$. The two values are not **exactly** equal.

✧ Technology Corner

Technology Note 1

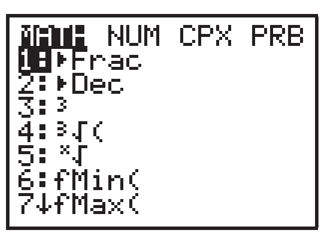
Example 1 on a Graphing Calculator

Begin by entering the two lines as $\mathtt{Y}_1$ and $\mathtt{Y}_2$. Choose a window where the intersection point is visible. To find the exact value of the intersection, go to `CALC` (via `2nd` `TRACE`) and choose `5:intersect`. You will be prompted to select the lines. Press `ENTER` for "`First curve?`", "`Second curve?`", and "`Guess?`". The intersection point will be displayed as in Screen 1.7. To avoid rounding errors, the intersection point must be converted to a fraction. To do this, `QUIT` to the home screen using `2ND` and `MODE`. Then press `X,T,θ,n`, then the `MATH` button, as shown in Screen 1.8. Choose `1:▷Frac` and then `ENTER` to convert the x-value of the intersection to a fraction, see Screen 1.9. To convert the y-value to a fraction, press `ALPHA` then `1` to get the variable `Y`. Next the `MATH` and `1:▷Frac` to see `Y` as a fraction.

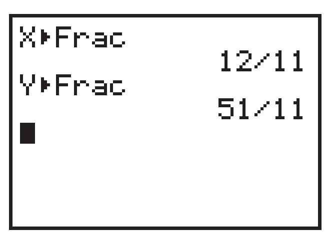


Screen 1.7

[-1, 3]×[-1, 10]



Screen 1.8

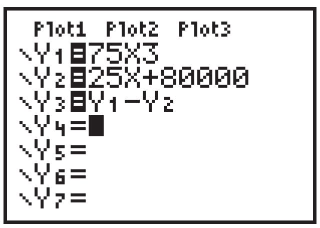


Screen 1.9

Technology Note 2 Example 2 on a Graphing Calculator

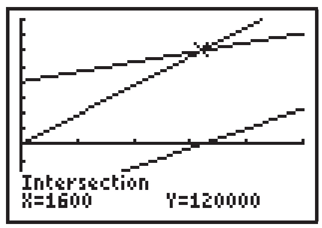
You can find the break-even quantity in Example 2 on your graphing calculator by finding where $P = 0$ or by finding where $C = R$. Begin by entering the revenue and cost equations into $\texttt{Y}_1$ and $\texttt{Y}_2$. You can subtract these two on paper to find the profit equation or have the calculator find the difference, as shown in Screen 1.10. To access the names $\texttt{Y}_1$ and $\texttt{Y}_2$, press the `VARS` , then right arrow to `Y-VARS` and `ENTER` to select `1:Function` then choose $\texttt{Y}_1$ or $\texttt{Y}_2$, as needed.

The intersection can be found in the same manner as Technology Note 1. To find where the profit is zero, return to the `CALC` menu and choose `2:zero`. Note you will initially be on the line $\texttt{Y}_1$. Use the down arrow twice to be on $\texttt{Y}_3$. Then use the left or right arrows to move to the left side of the zero of $\texttt{Y}_3$ and hit `ENTER` to answer the question "`Left Bound?`". Right arrow over to the right side of the place where $\texttt{Y}_3$ crosses the x-axis and hit `ENTER` to answer the question "`Right Bound?`". Place your cursor between these two spots and press `ENTER` to answer the last question, "`Guess?`". The result is shown in Screen 1.12.

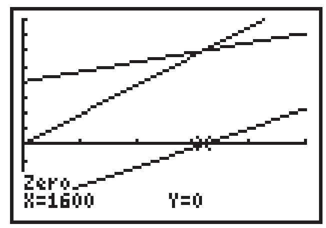


Screen 1.10

[0,2500] × [-80000,160000]



Screen 1.11



Screen 1.12

Self-Help Exercises 1.2

1. Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical medium-sized plant they estimated fixed costs at \$400,000 and estimated that it cost \$200 to produce each ton of fertilizer. The plant sells its fertilizer output at \$250 per ton. Find the break-even point. (Refer to Self-Help Exercise 1 in Section 1.1.)

Source: Rogers and Akridge 1996

2. The excess supply and demand curves for wheat worldwide were estimated by Schmitz and coworkers to be

Supply: $p = S(x) = 7x - 400$

Demand: $p = D(x) = 510 - 3.5x$

where p is price per metric ton and x is in millions of metric tons. Excess demand refers to the excess of wheat that producer countries have over their own consumption. Graph and find the equilibrium price and equilibrium quantity.

Source: Schmitz, Sigurdson, and Doering 1986

1.2 Exercises

Exercises 1 through 4 show linear cost and revenue equations. Find the break-even quantity.

1. $C = 2x + 4, \quad R = 4x$

2. $C = 3x + 10, \quad R = 6x$

3. $C = 0.1x + 2, \quad R = 0.2x$

4. $C = 0.03x + 1, \quad R = 0.04x$

In Exercises 5 through 8 you are given a demand equation and a supply equation. Sketch the demand and supply curves, and find the equilibrium point.

5. Demand: $p = -x + 6$, supply: $p = x + 3$

6. Demand: $p = -3x + 12$, supply: $p = 2x + 5$

7. Demand: $p = -10x + 25$, supply: $p = 5x + 10$

8. Demand: $p = -0.1x + 2$, supply: $p = 0.2x + 1$

Applications

9. Break-Even for Purses A firm has weekly fixed costs of \$40,000 associated with the manufacture of purses that cost \$15 per purse to produce. The firm sells all the purses it produces at \$35 per purse. Find the cost, revenue, and profit equations. Find the break-even quantity.

10. Break-Even for Lawn Mowers A firm has fixed costs of \$1,000,000 associated with the manufacture of lawn mowers that cost \$200 per mower to produce. The firm sells all the mowers it produces at \$300 each. Find the cost, revenue, and profit equations. Find the break-even quantity.

11. Rent or Buy Decision Analysis A forester has the need to cut many trees and to chip the branches. On the one hand he could, when needed, rent a large wood chipper to chip branches and logs up to 12 inches in diameter for \$320 a day. He estimates that his crew would use the chipper exactly 8 hours each day of rental use. Since he has a large amount of work to do, he is considering purchasing a new 12-inch wood chipper for \$28,000. He estimates that he will need to spend \$40 on maintenance per every 8 hours of use.

a. Let x be the number of hours he will use a wood chipper. Write a formula that gives him the total cost of renting for x hours.

b. Write a formula that gives him the total cost of buying and maintaining the wood chipper for x days of use.

c. If the forester estimates he will need to use the chipper for 1000 hours, should he buy or rent?

d. Determine the number of hours of use before the forester can save as much money by buying the chipper as opposed to renting.

12. Decision Analysis for Making Copies At Lincoln Library there are two ways to pay for copying. You can pay 5 cents a copy, or you can buy a plastic card for \$5 and then pay 3 cents a copy. Let x be the number of copies you make. Write an equation for your costs for each way of paying.

a. How many copies do you need to make before buying the plastic card is the same as cash?

b. If you wish to make 300 copies, which way of paying has the least cost?

13. Energy Decision Analysis Many home and business owners in northern Ohio can successfully drill for natural gas on their property. They are faced with the choice of obtaining natural gas free from their own gas well or buying the gas from a utility company. A garden center determines that they will need to buy \$5000 worth of gas each year from the local utility company to heat their greenhouses. They determine that the cost of drilling a small commercial gas well for the garden center will be \$40,000 and they assume that their well will need \$1000 of maintenance each year.

a. Write a formula that gives the cost of the natural gas bought from the utility for x years.

b. Write a formula that gives the cost of obtaining the natural gas from their well over x years.

c. How many years will it be before the cost of gas from the utility equals the cost of gas from the well?

CONNECTION: We know an individual living in a private home in northern Ohio who had a gas and oil well drilled some years ago. The well yields both natural gas and oil. Both products go into a splitter that separates the natural gas and the oil. The oil goes into a large tank and is sold to a local utility. The natural gas is used to heat the home and the excess is fed into the utility company pipes, where it is measured and purchased by the utility.

14. **Compensation Decision Analysis** A salesman for carpets has been offered two possible compensation plans. The first offers him a monthly salary of \$2000 plus a royalty of 10% of the total dollar amount of sales he makes. The second offers him a monthly salary of \$1000 plus a royalty of 20% of the total dollar amount of sales he makes.

    a. Write a formula that gives each compensation package as a function of the dollar amount x of sales he makes.

    b. Suppose he believes he can sell \$15,000 of carpeting each month. Which compensation package should he choose?

    c. How much carpeting will he sell each month if he earns the same amount of money with either compensation package?

15. **Truck Rental Decision Analysis** A builder needs to rent a dump truck from Acme Rental for a day for \$75 plus \$0.40 per mile and the same one from Bell Rental for \$105 plus \$0.25 per mile. Find a cost function for using each rental firm.

    a. Find the number of hours for which each cost function will give the same cost.

    b. If the builder wants to rent a dump truck for 150 days, which rental place will cost less?

16. **Wood Chipper Rental Decision Analysis** A contractor wants to rent a wood chipper from Acme Rental for a day for \$150 plus \$10 per hour or from Bell Rental for a day for \$165 plus \$7 per hour. Find a cost function for using each rental firm.

    a. Find the number of hours for which each cost function will give the same cost.

    b. If the contractor wants to rent the chipper for 8 hours, which rental place will cost less?

17. **Make or Buy Decision** A company includes a manual with each piece of software it sells and is trying to decide whether to contract with an outside supplier to produce it or to produce it in-house. The lowest bid of any outside supplier is \$0.75 per manual. The company estimates that producing the manuals in-house will require fixed costs of \$10,000 and variable costs of \$0.50 per manual. Find the number of manuals resulting in the same cost for contracting with the outside supplier and producing in-house.. If 50,000 manuals are needed, should the company go with outside supplier or go in-house?

18. **Decision Analysis for a Sewing Machine** A shirt manufacturer is considering purchasing a standard sewing machine for \$91,000 and for which it will cost \$2 to sew each of their standard shirts. They are also considering purchasing a more efficient sewing machine for \$100,000 and for which it will cost \$1.25 to sew each of their standard shirts. Find a cost function for purchasing and using each machine.

    a. Find the number of hours for which each cost function will give the same cost.

    b. If the manufacturer wishes to sew 10,000 shirts, which machine should they purchase?

19. **Equilibrium Point for Organic Carrots** A farmer will supply 8 bunches of organic carrots to a restaurant at a price of \$2.50 per bunch. If he can get \$0.25 more per bunch, he will supply 10 bunches. The restaurant's demand for organic carrot bunches is given by $p = D(x) = -0.1x+6$. What is the equilibrium point?

20. **Equilibrium Point for Cinnamon Rolls** A baker will supply 16 jumbo cinnamon rolls to a cafe at a price of \$1.70 each. If she is offered \$1.50, then she will supply 4 fewer rolls to the cafe. The cafe's demand for jumbo cinnamon rolls is given by $p = D(x) = -0.16x + 7.2$. What is the equilibrium point?

Referenced Applications

21. **Break-Even Quantity in Rice Production** Kekhora and McCann estimated a cost function for the rice production function in Thailand. They gave the fixed costs per hectare of \$75 and the variable costs per hectare of \$371. The revenue per hectare was given as \$573. Suppose the price for rice went down. What would be the

minimum price to charge per hectare to determine the break-even quantity?

Source: Kekhora and McCann 2003

22. Break-Even Quantity in Shrimp Production Kekhora and McCann estimated a cost function for a shrimp production function in Thailand. They gave the fixed costs per hectare of $1838 and the variable costs per hectare of $14,183. The revenue per hectare was given as $26,022. Suppose the price for shrimp went down. What would be the revenue to determine the break-even quantity?

Source: Kekhora and McCann 2003

23. Break-Even Quantity for Small Fertilizer Plants In 1996 Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical small-sized plant they estimated fixed costs at $235,487 and estimated that it cost $206.68 to produce each ton of fertilizer. The plant sells its fertilizer output at $266.67 per ton. Find the break-even quantity.

Source: Rogers and Akridge 1996

24. Break-Even Quantity for Large Fertilizer Plants In 1996 Rogers and Akridge of Purdue University studied fertilizer plants in Indiana. For a typical large-sized plant they estimated fixed costs at $447,917 and estimated that it cost $209.03 to produce each ton of fertilizer. The plant sells its fertilizer output at $266.67 per ton. Find the break-even quantity.

Source: Rogers and Akridge 1996

25. Break-Even Quantity on Kansas Beef Cow Farms Featherstone and coauthors studied 195 Kansas beef cow farms. The average fixed and variable costs are in the following table.

Variable and Fixed Costs

Costs per cow	
Feed costs	$261
Labor costs	$82
Utilities and fuel costs	$19
Veterinary expenses costs	$13
Miscellaneous costs	$18
Total variable costs	$393
Total fixed costs	$13,386

The farm can sell each cow for $470. Find the break-even quantity.

Source: Featherstone, Langemeier, and Ismet 1997

In Exercises 26 and 27 use the following information. In the Saur study of fenders mentioned in Exercise 41 of Section 1.1, the amount of energy consumed by each type of fender was also analyzed. The total energy was the sum of the energy needed for production plus the energy consumed by the vehicle used in carrying the fenders. If x is the miles traveled, then the total energy consumption equations for steel, aluminum, and NPN were as follows:

Steel: $E = 225 + 0.012x$
Al: $E = 550 + 0.007x$
NPN: $E = 565 + 0.007x$

26. Find the number of miles traveled for which the total energy consumed is the same for steel and NPN fenders. If 6000 miles is traveled, which material would use the least energy?

27. Find the number of miles traveled for which the total energy consumed is the same for steel and aluminum fenders. If 5000 miles is traveled, which material would use the least energy?

For Exercises 28 and 29 refer to the following information. In the Saur study of fenders mentioned in Example 3, the amount of CO_2 emissions in kg per 2 fenders of the production and utilization into the air of each type of fender was also analyzed. The total CO_2 emissions was the sum of the emissions from production plus the emissions from the vehicle used to carry the fenders. If x is the miles traveled, then the total CO_2 emission equations for steel, aluminum, and NPN were as follows:

Steel: $CO_2 = 21 + 0.00085x$
Aluminum: $CO_2 = 43 + 0.00045x$
NPN: $CO_2 = 23 + 0.00080x$

28. Find the number of pairs of fenders for which the total CO_2 emissions is the same for both steel and aluminum fenders. If 60,000 miles are traveled, which material would yield the least $CO2_2$

29. Find the number of pairs of fenders for which the total CO_2 emissions is the same for both steel and NPN fenders. If 30,000 miles are traveled, which material would yield the least CO_2?

30. Supply and Demand for Milk Demand and supply equations for milk were given by Tauer. In this paper he estimated demand and supply equations for bovine somatotropin-produced milk. The demand equation is $p = 55.9867 -$

$0.3249x$, and the supply equation is $p = 0.07958x$, where again p is the price in dollars per hundred pounds and x is the amount of milk measured in billions of pounds. Find the equilibrium point.

Source: Tauer 1994

31. Facility Location A company is trying to decide whether to locate a new plant in Houston or Boston. Information on the two possible locations is given in the following table. The initial investment is in land, buildings, and equipment.

	Houston	Boston
Variable cost	\$0.25 per item	\$0.22 per item
Annual fixed costs	\$4,000,000	\$4,210,000
Initial investment	\$16,000,000	\$20,000,000

a. Suppose 10,000,000 items are produced each year. Find which city has the lower annual total costs, not counting the initial investment.

b. Which city has the lower total cost over five years, counting the initial investment?

32. Facility Location Use the information found in the previous exercise.

a. Determine which city has the lower total cost over five years, counting the initial investment if 10,000,000 items are produced each year.

b. Find the number of items yielding the same cost for each city counting the initial investment.

Extensions

For Exercises 33 through 36 consider the following study. As mentioned in Example 3 Saur and colleagues did a careful study of the cost of manufacturing automobile fenders using five different materials: steel, aluminum, and three injection-molded polymer blends: rubber-modified polypropylene (RMP), nylon-polyphenylene oxide (NPN), and polycarbonate-polybutylene terephthalate (PPT). The following table gives the fixed and variable costs of manufacturing each pair of fenders. Note that only an integer number of pairs of fenders can be counted.

Variable and Fixed Costs of Pairs of Fenders

Costs	Steel	Aluminum	RMP	NPN	PPT
Variable	\$5.26	\$12.67	\$13.19	\$9.53	\$12.55
Fixed	\$260,000	\$385,000	\$95,000	\$95,000	\$95,000

33. How many pairs of fenders are required for the cost of the aluminum ones to equal the cost of the RMP ones?

34. How many pairs of fenders are required for the cost of the steel ones to equal the cost of the NPN ones?

35. How many pairs of fenders are required for the cost of the steel ones to equal the cost of the PPT ones?

36. How many pairs of fenders are required for the cost of the steel ones to equal the cost of the RMP ones?

37. Process Selection and Capacity A machine shop needs to drill holes in a certain plate. An inexpensive manual drill press could be purchased that will require large labor costs to operate, or an expensive automatic press can be purchased that will require small labor costs to operate. The following table summarizes the options.

Machine	Annual Fixed Costs	Variable Labor Costs	Production plates/hour
Manual	\$1000	\$16.00/hour	10
Automatic	\$8000	\$2.00/hour	100

Suppose these are the only fixed and variable costs.

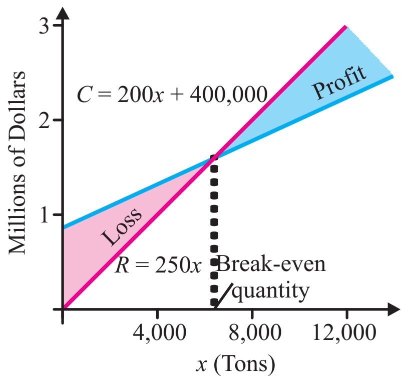
a. If x is the number of plates produced per hour, find the total cost using the manual drill press per hour and the cost function using the automatic drill press.

b. Find the number of plates produced per hour for which the manual and automatic drill presses will cost the same.

38. Decision Analysis Roberts formulated a mathematical model of corn yield response to nitrogen fertilizer in high-yield response land and low-yield response land. They estimated a profit equation $P = f(N)$ that depended only on the number of pounds of nitrogen fertilizer per acre used. For the high-yield response land they estimated that $P = H(N) = 0.17N + 96.6$ and for the low-yield response land they estimated that $P = L(N) = 0.48N + 26.0$. A farmer has both types of land in two separate fields but does not have the time to use both fields. How much nitrogen will result in each response land yielding the same profit? Which field should be selected if 250 pounds of nitrogen is used?

Source: Roberts, English, and Mahajashetti 2000

Solutions to Self-Help Exercises 1.2



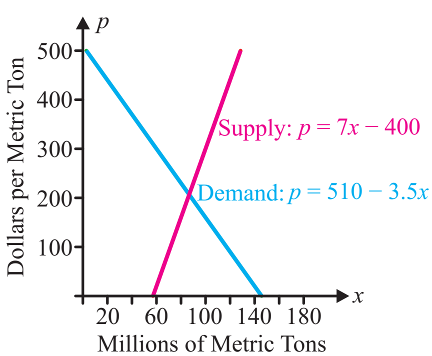
1. Let x be the number of tons of fertilizer produced and sold. Then the cost and revenue equations are

$$\begin{aligned} C(x) &= (\text{variable cost}) + (\text{fixed cost}) \\ &= 200x + 400,000 \\ R(x) &= (\text{price per ton}) \times (\text{number of tons sold}) \\ &= 250x \end{aligned}$$

The cost and revenue equations are graphed in the figure. To find the break-even quantity set $C(x) = R(x)$ and solve for x:

$$\begin{aligned} R(x) &= C(x) \\ 250x &= 200x + 400,000 \\ 50x &= 400,000 \\ x &= 8000 \end{aligned}$$

Thus, the plant needs to produce and sell 8000 tons of fertilizer to break-even (i.e., for profits to be zero).



2. The graphs are shown in the figure. To find the equilibrium price, set $D(p) = S(p)$ and obtain

$$\begin{aligned} D(p) &= S(p) \\ 510 - 3.5x &= 7x - 400 \\ 10.5x &= 910 \\ x &\approx 86.7 \end{aligned}$$

With $x = 86.7$, $p = D(86.7) = 510 - 3.5(86.7) \approx 207$. The equilibrium price is \$207 per metric ton, and the equilibrium quantity is 86.7 million metric tons.

1.3 Gauss Elimination for Systems of Linear Equations

In the previous section we considered systems of linear equations that had two variables and two equations. In this section we look at larger systems of linear equations that have unique solutions. Systems with no solution or non-unique solutions are covered in Section 1.4. To solve these larger systems we will develop the method of Gauss elimination, an efficient and systematic manner of finding the solution to systems of linear equations. For example, consider the following system of linear equations:

$$\begin{aligned} x + 2y + 7z &= 14 \\ x - 3y + 2z &= 4 \\ 2x - y - 3z &= -5 \end{aligned}$$

Our method is to first **eliminate** the x variable from all the equations below the first. Then **eliminate** the y variable from the equation below the second, and so forth. After this, we are left with a system like the following:

$$\begin{aligned} x + 2y + 7z &= 14 \\ y - z &= 2 \\ z &= 1 \end{aligned}$$

We notice that we have found that $z = 1$. We now use **backward substitution**. That is, the value for z substituted into the next to the last equation. We can solve for the y variable and then substitute the values for y and z into the first equation and solve for x. In the discussion below we will determine an efficient and systematical manner of preparing the system for backward substitution.

✧ Creating Systems of Linear Equations

In the first example we examine how to translate English sentences into equations. Nearly all word problems follow a similar format. First you are given some information and then you are asked "How much?" or "How many?". Always begin your work on a word problem by clearly defining your variables — they are the answer to the "How much?" or "How many?" question you were asked. The next step is to take the given information and translate it into an equation using the variables you defined. A table or diagram is often helpful to organize the information.

EXAMPLE 1 **Document Scheduling** An insurance company has two types of documents to process: contracts and leases. Each contract needs to be examined for 2 hours by the accountant and 3 hours by the attorney, while each lease needs to be examined for 4 hours by the accountant and 1 hour by the attorney. If the accountant has 40 hours and the attorney 30 hours each week to spend working on these documents, how many documents of each type can they process?

Solution The question asked is "how many documents of each type?". This indicates what the unknowns are: the number of contracts and the number of leases. So let

$$\begin{aligned} x &= \text{the number of contracts} \\ y &= \text{the number of leases} \end{aligned}$$

Since there are x contracts and the accountant spends 2 hours per week on each of these, the accountant spends $2x$ hours per week on contracts. The attorney spends $3x$ hours per week on contracts. There are y leases and the accountant spends 4 hours per week on each of these, so the accountant spends $4y$ hours per week on leases and the attorney spends y hours per week on leases.

Let us now create a table that summarizes all of the given information. When we organize information in a table note that we have written the conditions we must obey in a row. The first row in the table summarizes in mathematical notation how the accountant spends her time and the second row summarizes how the attorney spends his time.

	Contracts	Leases	Total
Accountant time	$2x$	$4y$	40 hours
Attorney time	$3x$	y	30 hours

The accountant spends $2x+4y$ hours on these two documents and this must equal the total available accountant time which is 40. So the first equation must be

$$2x + 4y = 40$$

The attorney spends $3x + y$ hours on these two documents and this must equal the total available attorney time which is 30. So the second equation must be

$$3x + y = 30$$

We then have the system of equations

$$\begin{aligned} 2x + 4y &= 40 \text{ accountant hours} \\ 3x + y &= 30 \text{ attorney hours} \end{aligned}$$

The solution of this system will then tell us the number contracts and leases processed per week. ✦

✧ Gauss Elimination With Two Equations

Let us now solve this system we found in Example 1. There are several basic ways of solving such a system. First, we could solve each equation for y and have $y = -(1/2)x + 10$ and $y = -3x + 30$. As in the last section we can set $-(1/2)x + 10 = -3x + 30$ and solve for x. This method will only work for systems with two variables.

Another basic way of solving this system of equations is to use substitution. For example, we could solve for y in the second equation and obtain $y = 30 - 3x$. Then substitute this into the first equation to obtain $2x + 4(30 - 3x) = 40$. Now solve for x. We then find y, since $y = 30 - 3x$. The method works well when we have two equations with two unknowns, but becomes cumbersome and not very useful when applied to systems with more than two equations and two unknowns.

We need a systematic method that will always work, no matter how complicated the system. Such a method is **Gauss elimination** with backward substitution. This method will serve us well in subsequent sections. The strategy begins with eliminating x from the second equation. To make this easier, we wish to have the coefficient of x in the first equation equal to 1. To do this simply divide the first equation by 2 to get the system S1,

$$\begin{aligned} x + 2y &= 20 \\ 3x + y &= 30 \end{aligned} \tag{S1}$$

Now we more readily see how to eliminate x from the second equation. We multiply the first equation by -3, written -3(first), and add this to the second equation to get a new second equation. We have

$$\begin{array}{r} -3(\text{first}) : -3x - 6y = -60 \\ \underline{+(\text{second}) : \quad 3x + y = 30} \\ -5y = -30 \end{array}$$

This last equation is the new second equation. We now have the system as

$$\begin{aligned} x + 2y &= 20 \\ -5y &= -30 \end{aligned}$$

Notice that we have **eliminated** the x variable from the second equation. We then divide the last equation by -5 and obtain

$$\begin{aligned} x + 2y &= 20 \\ y &= 6 \end{aligned}$$

We have $y = 6$. Now substitute this back into the first equation and obtain

$$\begin{aligned} x + 2y &= 20 \\ x + 2(6) &= 20 \\ x &= 20 - 12 = 8 \end{aligned}$$

We then have the solution $(x, y) = (8, 6)$. This means that eight of the contracts documents can be processed and six of the leases.

When we manipulated the equations in Example 1, we followed three basic rules that allowed us to change the system of equations from one form to another form with the same solution as the original system. But it was easy to determine the solution in the new system. For convenience we will denote the first equation by E_1, the second equation by E_2, the third equation by E_3, and so forth. The rules that must be followed are below.

Elementary Equation Operations

1. Two equations can be interchanged, $E_i \leftrightarrow E_j$.
2. An equation may be multiplied by a non-zero constant, $kE_i \rightarrow E_i$.
3. A multiple of one equation may be added to another equation, $E_i + kE_j \rightarrow E_i$.

Gauss elimination is the systematic use of these three allowed operations to put the system of equations into an easily solved form.

✧ Gauss Elimination

We will now develop Gauss elimination for a system with any number of equations with any number of unknowns. In this section we continue to restrict ourselves to examples for which there is a unique solution. We will consider the case when there is no solution or are infinitely many solutions in the next section. First we solve a system of three equations with three unknowns.

EXAMPLE 2 **Scheduling Shirts** A firm produces three products, a dress shirt, a casual shirt, and a sport shirt, and uses a cutting machine, a sewing machine, and a packaging machine in the process. To produce each gross[1] of dress shirts requires 3 hours on the cutting machine and 2 hours on the sewing machine. To produce each gross of casual shirts requires 5 hours on the cutting machine and 1 hour on the sewing machine. To produce each gross of sport shirts requires 7 hours on the cutting machine and 3 hours on the sewing machine. It takes 2 hours to package each gross of each type of shirt. If the cutting machine is available for 480 hours, the sewing machine for 170 hours, and the packaging machine for 200 hours, how many gross of each type of shirt should be produced to use all of the available time on the three machines?

Solution We are asked "how many gross of each type of shirt should be produced?" So begin by defining the variables. Let

$x =$ the number of gross of dress shirts made,
$y =$ the number of gross of casual shirts made, and
$z =$ the number of gross of sport shirts made.

We then create a table that summarizes all of the given information. Once again we will have each row of the table represent how a condition is fulfilled. The first row shows how the time is used on the packaging machine, the second shows the time on the cutting machine, and the third row shows the time used on the sewing machine. Organizing the information in this way will make it easier to form the equations needed to answer the question.

	Dress Shirts	Casual Shirts	Sport Shirts	Total Hours Available
Hours on packaging machine	$2x$	$2y$	$2z$	200
Hours on cutting machine	$3x$	$5y$	$7z$	480
Hours on sewing machine	$2x$	y	$3z$	170

Since each gross of each style of shirt requires 2 hours on the packaging machine, and this machine is available for 200 hours, we must have

$$2x + 2y + 2z = 200$$

[1] A gross is a dozen dozen, or 144.

Since the number of hours on the cutting machine is $3x+5y+7z$, while the total hours available is 480, we must have

$$3x + 5y + 7z = 480$$

Looking at the time spent and available on the sewing machine gives

$$2x + y + 3z = 170$$

Together, these three equations gives the system of linear equations

$$\begin{aligned} 2x + 2y + 2z &= 200 \text{ packaging machine hours} \\ 3x + 5y + 7z &= 480 \text{ cutting machine hours} \\ 2x + y + 3z &= 170 \text{ sewing machine hours} \end{aligned}$$

Begin by dividing the first equation by 2 so that the coefficient of x in the first equation is 1. Doing this gives the system

$$\begin{cases} E_1 : x + y + z = 100 \\ E_2 : 3x + 5y + 7z = 480 \\ E_3 : 2x + y + 3z = 170 \end{cases}$$

We need to use the first equation to eliminate x from the second and third equations. We do this one equation at a time. Start by using the first equation to eliminate x from the second equation. We can do this by multiplying -3 times the first equation and add this to the second equation to become our new second equation. That is, $E_2 - 3E_1 \to E_2$. We have

$$\begin{array}{r} -3E_1 : -3x - 3y - 3z = -300 \\ E_2 : 3x + 5y + 7z = 480 \\ \hline E_2 - 3E_1 : 2y + 4z = 180 \end{array}$$

Now we use the first equation to eliminate x from the third equation. We can do this by multiplying -2 times the first equation and add this to the third equation ($E_3 - 2E_1 \to E_3$) to form our new third equation. We have

$$\begin{array}{r} -2E_1 : -2x - 2y - 2z = -200 \\ E_3 : 2x + y + 3z = 170 \\ \hline E_3 - 2E_1 : - y + z = -30 \end{array}$$

Using the new second and third equations from above in the bracketed system we have

$$\begin{cases} E_1 : x + y + z = 100 \\ E_2 : 2y + 4z = 180 \\ E_3 : - y + z = -30 \end{cases}$$

Pause for a moment to see what we have accomplished. We started with a system of three equations with three unknowns. But if we just look at the new second and third equations,

$$\begin{cases} E_2 : 2y + 4z = 180 \\ E_3 : - y + z = -30 \end{cases}$$

we see that we now have *two* equations with the *two* unknowns y and z as we have eliminated x from the last two equations. This is much simpler and we solve these two equations as we did in Example 1, while, for now, leaving the first equation unchanged. Begin by dividing the second equation by 2 (that is, $\frac{1}{2}E_2 \to E_2$) to have a y-coefficient of 1 and obtain

$$\begin{cases} E_1 : x + y + z = 100 \\ E_2 : y + 2z = 90 \\ E_3 : - y + z = -30 \end{cases}$$

Now we use the second equation to eliminate the y variable from the third equation. We replace the third equation with the sum of the second and third equations ($E_2 + E_3 \to E_3$). We have

$$\begin{array}{rrcr} E_2: & y + 2z & = & 90 \\ E_3: & -\,y + z & = & -30 \\ \hline E_2 + E_3: & 3z & = & 60 \end{array}$$

This last equation is the new third equation. We now have

$$\begin{cases} E_1 : x + y + z = 100 \\ E_2 : y + 2z = 90 \\ E_3 : 3z = 60 \end{cases}$$

Finally, we divide the last equation by 3 ($\frac{1}{3}E_3 \to E_3$) and our system is

$$\begin{cases} E_1 : x + y + z = 100 \\ E_2 : y + 2z = 90 \\ E_3 : z = 20 \end{cases}$$

with the value $z = 20$. Now we backward substitute.

$$E_3 : \qquad z = 20$$

$$\begin{aligned} E_2 : \qquad y + 2z &= 90 \\ y + 2(20) &= 90 \\ y &= 50 \end{aligned}$$

$$\begin{aligned} E_3 : \qquad x + y + z &= 100 \\ x + (50) + (20) &= 100 \\ x &= 30 \end{aligned}$$

So the solution is $(x, y, z) = (30, 50, 20)$. Thus, the firm should produce 30 gross of dress shirts, 50 gross of casual shirts, and 20 gross of sport shirts to use all of the available time on the three machines. ✦

✧ Gauss Elimination Using the Augmented Matrix

Let us continue by solving the following system of equations.

$$\begin{aligned} x + 2y - 2z &= 1 \\ 2x + 7y + 2z &= -1 \\ x + 6y + 7z &= -3 \end{aligned}$$

It is convenient to write this system in the abbreviated form as

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 7 & 2 & -1 \\ 1 & 6 & 7 & -3 \end{array}\right]$$

Such a rectangular array of numbers is called a **matrix**. We refer to this matrix as the **augmented matrix** for the system. The only difference between this augmented matrix and the original system is that in the augmented matrix the symbols x, y, z, and $=$ have been dropped. This spares us the work of always writing these symbols for each step of the solution. Now if we wish to interchange two equations in the system, we interchange the two corresponding rows in the augmented matrix. If we wish to multiply a constant times each side of one equation, we multiply the constant times each member of the corresponding row, etc. In general we have the following.

Elementary Row Operations

1. Interchange the ith row with the jth row ($R_i \leftrightarrow R_j$).
2. Multiply each member of the ith row by a nonzero constant k ($kR_i \to R_i$).
3. Replace each element in the ith row with the corresponding element in the ith row plus k times the corresponding element in the jth row ($R_i + kR_j \to R_i$).

Notice that our notation using elementary row operations is similar to that for equations. So, for example, the row R_i corresponds to the equation E_i.

EXAMPLE 3 Using the Augmented Matrix Solve the system of linear equations using the augmented matrix.

$$\begin{cases} E_1: \quad x + 2y - 2z = 1 \\ E_2: 2x + 7y + 2z = -1 \\ E_3: \quad x + 6y + 7z = -3 \end{cases}$$

Solution As usual, we first want the coefficient of x in the first equation to be 1. This is the same as having a 1 in the first row and first column of the augmented matrix. This we already have. Now we wish to eliminate x from the second and third equations. This corresponds to wanting zeros as the first element in each of the second and third rows. We write each equation as a row and proceed as before, but with the augmented matrix and row operations.

Technology Option
Example 3 is solved using a graphing calculator in Technology Note 1 on page 44

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 2 & 7 & 2 & -1 \\ 1 & 6 & 7 & -3 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 3 & 6 & -3 \\ 0 & 4 & 9 & -4 \end{array}\right] \tfrac{1}{3}R_2 \to R_2$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 4 & 9 & -4 \end{array}\right] R_3 - 4R_2 \to R_3$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Notice that we have created a diagonal of 1's with zeros below each of the 1's in the matrix. This augmented matrix corresponds to a system of equations that is ready for backward substitution,

$$\begin{aligned} E_1 &: x + 2y - 2z = 1 \\ E_2 &: \quad y + 2z = -1 \\ E_3 &: \quad z = 0 \end{aligned}$$

This gives $z = 0$. Now backward substitute and obtain

$$\begin{aligned} E_2: \quad y + 2z &= -1 \\ y + 2(0) &= -1 \\ y &= -1 \end{aligned}$$

$$\begin{aligned} E_1: \quad x + 2y - 2z &= 1 \\ x + 2(-1) - 2(0) &= 1 \\ x &= 3 \end{aligned}$$

The solution is $(x, y, z) = (3, -1, 0)$. ✦

EXAMPLE 4 **Gauss Elimination With Four Equations** Solve the following system of equations.

$$\begin{cases} E_1: & 3x + 6y + 3z + 3u = 450 \\ E_2: & 2x + 6y + 4z + 4u = 500 \\ E_3: & \quad 3y + 5z + 7u = 480 \\ E_4: & \quad 2y + z + 3u = 170 \end{cases}$$

Solution The augmented matrix is

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 3 & 450 \\ 2 & 6 & 4 & 4 & 500 \\ 0 & 3 & 5 & 7 & 480 \\ 0 & 2 & 1 & 3 & 170 \end{array}\right]$$

Begin by making the coefficient of the first variable in the first row 1 by dividing the first row by 3,

$$\left[\begin{array}{cccc|c} 3 & 6 & 3 & 3 & 450 \\ 2 & 6 & 4 & 4 & 500 \\ 0 & 3 & 5 & 7 & 480 \\ 0 & 2 & 1 & 3 & 170 \end{array}\right] \tfrac{1}{3}R_1 \to R_1 \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 150 \\ 2 & 6 & 4 & 4 & 500 \\ 0 & 3 & 5 & 7 & 480 \\ 0 & 2 & 1 & 3 & 170 \end{array}\right]$$

We need to place a zero in the second row and first column. We have

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 150 \\ 2 & 6 & 4 & 4 & 500 \\ 0 & 3 & 5 & 7 & 480 \\ 0 & 2 & 1 & 3 & 170 \end{array}\right] \quad R_2 - 2R_1 \to R_2 \quad \longrightarrow \quad \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 150 \\ 0 & 2 & 2 & 2 & 200 \\ 0 & 3 & 5 & 7 & 480 \\ 0 & 2 & 1 & 3 & 170 \end{array}\right]$$

This corresponds to

$$\begin{cases} E_1: x + 2y + z + u = 150 \\ E_2: \phantom{x +{}} 2y + 2z \quad 2u = 200 \\ E_3: \phantom{x +{}} 3y + 5z + 7u = 480 \\ E_4: \phantom{x +{}} 2y + z + 3u = 170 \end{cases}$$

Notice how we began with a complicated system with four equations with four unknowns, and now the last three equations are just three equations with three unknowns.

$$\begin{cases} E_2: 2y + 2z \quad 2u = 200 \\ E_3: 3y + 5z + 7u = 480 \\ E_4: 2y + z + 3u = 170 \end{cases}$$

This is much simpler. As usual we ignore the first equation for now and just work with the last three. But notice that the last three equations are the same as the three equations we solved in Example 2! Only the variables are labeled differently. So we can use this observation and recalling the solution to Example 2 we see that $y = 30$, $z = 50$, and $u = 20$. Now we backward substitute this into E_1 and obtain

$$\begin{aligned} E_1: x + \quad 2y + \quad z + \quad u &= 150 \\ x + (30) + (50) + (20) &= 150 \\ x &= 50 \end{aligned}$$

The solution is $(x, y, z, u) = (50, 30, 50, 20)$. ✦

EXAMPLE 5 **Gauss Elimination With Three Equations** Solve the following system of equations.

$$\begin{aligned} y - 3z &= 1 \\ x + y - 2z &= 2 \\ x + y - z &= 1 \end{aligned}$$

Solution The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & 1 \\ 1 & 1 & -2 & 2 \\ 1 & 1 & -1 & 1 \end{array}\right]$$

Technology Option
Example 5 is solved using a graphing calculator in Technology Note 2 on page 45

Our normal procedure would be to use the first equation to eliminate x from the second equation. But since the coefficient of x in the first equation is **zero**, this is not possible. The simple solution is to switch the first equation with another equation that has a nonzero coefficient of x. In this case we may pick the second or the third equation. We pick the second. Then

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & 1 \\ 1 & 1 & -2 & 2 \\ 1 & 1 & -1 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 1 & -3 & 1 \\ 1 & 1 & -1 & 1 \end{array}\right] \quad R_3 - R_1 \to R_3$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

We now have an augmented matrix with 1's along the diagonal and zeros below each 1, and so we are ready to write out the system of equations. This augmented matrix corresponds to

$$\begin{aligned} E_1 &: x + y - 2z = 2 \\ E_2 &: \quad\;\; y - 3z = 1 \\ E_3 &: \qquad\quad\;\; z = -1 \end{aligned}$$

We have $z = -1$, then backward substitution gives

$$\begin{aligned} E_2 : \qquad y - 3z &= 1 \\ y - 3(-1) &= 1 \\ y &= 1 - 3 = -2 \end{aligned}$$

$$\begin{aligned} E_1 : \qquad x + y - 2z &= 2 \\ x + (-2) - 2(-1) &= 2 \\ x &= 2 + 2 - 2 = 2 \end{aligned}$$

So the solution is $(x, y, z) = (2, -2, -1)$. ✦

✧ Additional Applications

EXAMPLE 6 **Investment Allocation** A fund manager has been given a total of \$900,000 by a client to invest in certain types of stocks and bonds. The client requires that twice as much money be in bonds as in stocks. Also stocks must be restricted to a certain class of low-risk stocks and another class of medium-risk stocks. Furthermore, the client demands an annual return of 8%. The fund manager assumes from historical data that the low-risk stocks should return 9% annually, the medium-risk stocks 11% annually, and the bonds 7%. How should the fund manager allocate funds among the three groups to meet all the demands of the client?

Solution Begin by defining the variables needed to answer the question. Let
x = the amount of money in dollars allocated to low-risk stocks,
y = the amount of money in dollars allocated to medium-risk stocks, and
z = the amount of money in dollars allocated to bonds.

The total invested is $x+y+z$ dollars and this must equal 900,000. Thus, our first equation is

$$x + y + z = 900,000$$

Low-risk stocks will earn 9% per year, so $0.09x$ is the number of dollars low-risk stocks will return in one year. Medium-risk will earn 11% per year, so $0.11y$ is the number of dollars medium-risk stocks will return in one year. Bonds will earn 7% per year, so $0.07z$ is the number of dollars bonds will return in one year. These three returns must add to the return the client has demanded, which is $0.08 \times 900,000 = 72,000$. Thus,

$$0.07x + 0.11y + 0.07z = 72,000$$

Finally, we must have twice as much money in bonds as in stocks. The amount we have in stocks is $x + y$ and the amount in bonds is z. To see this ratio we can look at Figure 1.19.

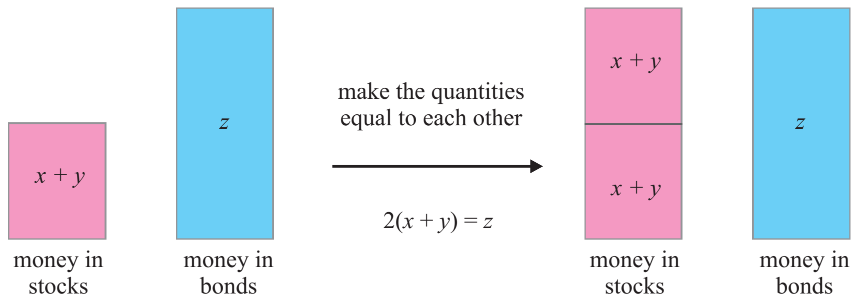


Figure 1.19

Or we can organize our information in a table and look for a pattern in how the variables are related to each other as shown in the table below.

$x+y$	1	2	3	...	$x+y$
z	2	4	6	...	$2(x+y)$

Technology Option
Example 6 is solved using a graphing calculator in Technology Note 3 on page 46

Therefore, we must have $z = 2(x + y)$. For Gauss Elimination the variables must be on the left, so write this equation as

$$2x + 2y - z = 0$$

Our three equations are then

$$\begin{array}{rrrl} x + & y + & z = & 900,000 \text{ total dollars invested} \\ 0.09x + & 0.11y + & 0.07z = & 72,000 \text{ total return} \\ 2x + & 2y - & z = & 0 \text{ ratio of stocks to bonds} \end{array}$$

The augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 900,000 \\ 0.09 & 0.11 & 0.07 & 72,000 \\ 2 & 2 & -1 & 0 \end{array}\right]$$

Proceeding with the Gauss elimination we have

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 900,000 \\ 0.09 & 0.11 & 0.07 & 72,000 \\ 2 & 2 & -1 & 0 \end{array}\right] \begin{array}{l} R_2 - 0.09R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array} \longrightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 900,000 \\ 0 & 0.02 & -0.02 & -9000 \\ 0 & 0 & -3 & -1,800,000 \end{array}\right] \begin{array}{l} 50R_2 \to R_2 \\ -\frac{1}{3}R_3 \to R_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 900,000 \\ 0 & 1 & -1 & -450,000 \\ 0 & 0 & 1 & 600,000 \end{array}\right]$$

This corresponds to

$$\begin{aligned} x + y + z &= 900,000 \\ y - z &= -450,000 \\ z &= 600,000 \end{aligned}$$

Backward substituting gives

$$\begin{aligned} E_3: \quad z &= 600,000 \\[1em] E_2: \quad y - z &= -450,000 \\ y - 600,000 &= -450,000 \\ y &= 150,000 \\[1em] E_1: \quad x + y + z &= 900,000 \\ x + 150,000 + 600,000 &= 900,000 \\ x &= 150,000 \end{aligned}$$

The client places \$150,000 in each type of stock and \$600,000 in bonds. ✦

✧ Introduction to Gauss-Jordan (Optional)

Recall in Example 3 that we ended with the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

which we then solved by backward substitution.

It is possible, however, to continue to perform row operations to reduce this matrix to a simple form that allows us to read off the solution. Keep in mind, however, that the amount of calculations that we will do is no more nor less than what we did in backward substitution.

We start with the one that is circled in the first augmented matrix below. We use this one and the row it is in to perform row operations with the goal of obtaining zeros in the column above the circled one. We repeat this procedure using the circled one in the second augmented matrix.

Technology Option
Example 3 is solved using a graphing calculator in Technology Note 4 on page 46

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & \textcircled{1} & 0 \end{array}\right] \begin{array}{l} R_1 + 2R_3 \to R_1 \\ R_2 - 2R_3 \to R_2 \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & \textcircled{1} & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right] R_1 - 2R_2 \to R_1$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

This can then be written as the following system.

$$\begin{aligned} x \quad\quad &= 3 \\ y \quad &= -1 \\ z &= 0 \end{aligned}$$

As before, we have $z = 0$. But now we see that we immediately have $y = -1$ and $x = 3$. The solution is then $(x, y, z) = (3, -1, 0)$ as before.

We outline how the Gauss-Jordan method could be used on the following augmented matrix.

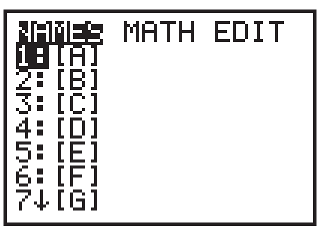
$$\left[\begin{array}{cccc|c} 1 & x & x & x & x \\ 0 & 1 & x & x & x \\ 0 & 0 & 1 & x & x \\ 0 & 0 & 0 & 1 & d \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & x & x & 0 & x \\ 0 & 1 & x & 0 & x \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & x & 0 & 0 & x \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{array}\right]$$

So, the solution is seen to be (a, b, c, d).

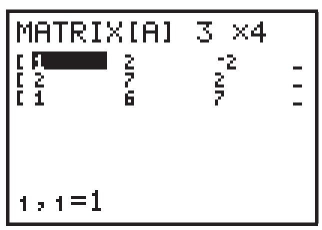
✧ Technology Corner

Technology Note 1 — Example 3 on a Graphing Calculator

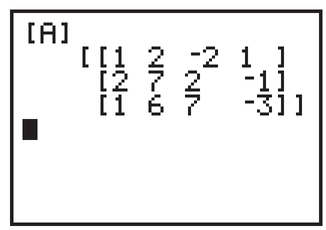
Begin by entering the matrix in the calculator. Matrices are accessed via the `MATRX` button on the TI-83. On the TI-83Plus and all TI-84s, the matrix commands are accessed with the `2ND` and `x⁻¹` buttons. Either way, you will see Screen 1.13 for the matrix commands. To enter the matrix, right arrow to the `EDIT` command and press `ENTER` to edit matrix `[A]`. Now you are in the matrix edit menu. Enter 3 and 4 from the keypad to create a 3x4 matrix and then proceed to enter the matrix elements, as shown in Screen 1.14. To return to the home screen and see the matrix, do `2ND` `MODE` to quit, then access the matrix commands and choose `NAMES` and `ENTER` to paste the matrix name `[A]` on the home screen. Press `ENTER` again to see the matrix displayed as it is in Screen 1.15.



Screen 1.13



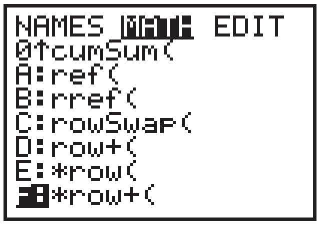
Screen 1.14



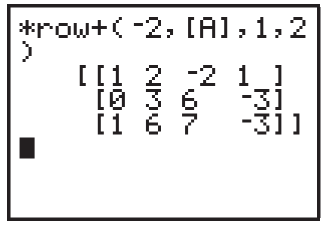
Screen 1.15

Now to access the row commands, return to the matrix commands and right arrow to `MATH`. Next, use the down arrow until the row operations are displayed as in Screen 1.16. To do the operation $R_2 - 2R_1 \rightarrow R_2$, choose the command `F:*row+(` and enter to return to the homescreen with the `*row+(` command displayed. The arguments for this command are the value to multiply by (here it is -2), the matrix to use (here it is `[A]`, which is accessed via the `MATRIX NAME` command), the row number that will be multiplied (here it is row 1) and then the row this multiplied row is to be added to (here it is row 2). Press enter and the result is displayed in Screen 1.17. This matrix can be stored or simply used in the next operation. To store this result as matrix `[B]`, press the `STO` and then `MATRIX NAME` and choose `2:[B]`.

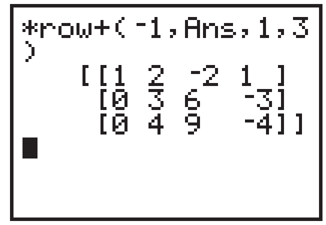
Since we often carry out a series of row operations, we can use the `Ans` feature to access the last matrix used. In the next operation, $R_3 - R_1 \rightarrow R_3$, we will access the previous matrix by pressing 2ND and then (-) to have Ans for our matrix name. See Screen 1.18.



Screen 1.16

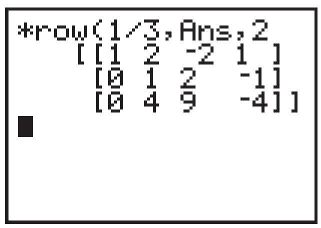


Screen 1.17

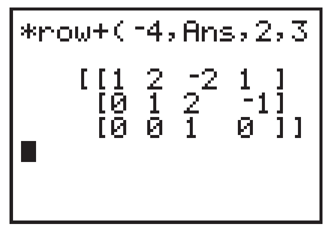


Screen 1.18

The next command is to do $\frac{1}{3}R_2 \rightarrow R_2$. The command `*row(` will do the row multiplication and it is found under the MATRIX MATH menu. The arguments for this command are the value to multiply the row by, the name of the matrix and the row to be multiplied. As before, we want to carry out this operation on the previous matrix, as shown in Screen 1.19. The last step, $R_3 - 4R_2 \rightarrow R_3$ is shown in Screen 1.20.



Screen 1.19



Screen 1.20

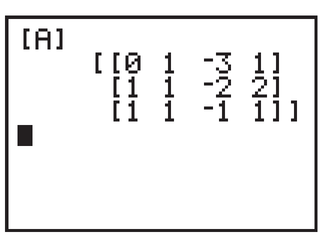
The use of these matrix commands are summarized in Table 1.2. It is assumed that matrix [A] is the augmented matrix to be operated on.

Elementary Row Operation	Calculator Command
$R_i \leftrightarrow R_j$	`rowSwap([A], i, j)`
$kR_i \rightarrow R_i$	`*row(k, [A], i)`
$R_i + kR_j \rightarrow R_i$	`*row+(k, [A], i, j)`

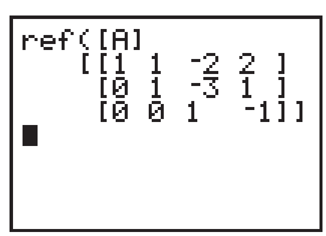
Table 1.2

Ⓣ Technology Note 2 Example 5 on a Graphing Calculator

Begin by entering the augmented matrix into the calculator as above (see Screen 1.21). Then return to the MATRIX MATH menu and scroll to `A:ref(` and ENTER . Next use MATRIX NAME to enter the name of the matrix where the values were entered. Then ENTER to do the `ref(` operation, as shown in Screen 1.22. The matrix is now ready for backward substitution.



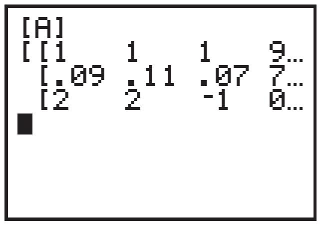
Screen 1.21



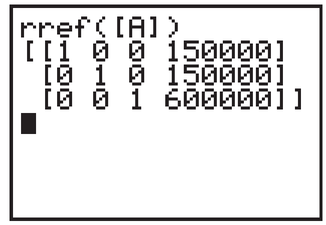
Screen 1.22

Technology Note 3 **Example 6 on a Graphing Calculator**

The TI calculators are able to carry out the backward substitution within the augmented matrix. The explanation for this is detailed in the optional Gauss-Jordan subsection. However, the method is very straightforward to implement on the calculator. Begin by entering the augmented matrix into the calculator, as shown in Screen 1.23. Next go to the `MATRIX MATH` menu and scroll to `B:rref(` and `ENTER`. Next use `MATRIX NAME` to enter the name of the matrix where the values were entered. Then `ENTER` to do the `rref(` operation, as shown in Screen 1.24. The answer can be found by writing each row as an equation.

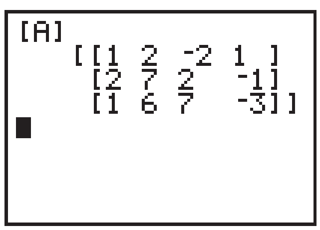


Screen 1.23

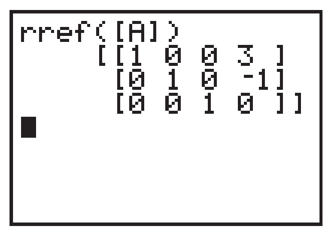


Screen 1.24

Technology Note 4 **Example 3 on a Graphing Calculator** The `rref(` command outlined in Technology Note 3 is in fact, using the Gauss-Jordan method on the augmented matrix. As long as the number of rows in the augmented matrix is greater than or equal to 1+ the number of columns in the matrix, the calculator can perform all the row operations to obtain the solution to the system of linear equations. See Screens 1.25 and 1.26 for this example.



Screen 1.25



Screen 1.26

Self-Help Exercises 1.3

1. A store sells 30 sweaters on a certain day. The sweaters come in three styles: A, B, and C. Style A costs \$30, style B costs \$40, and style C costs \$50, If the store sold \$1340 worth of these sweaters on that day and the number of style C sold exceeded by 6 the sum of the other two styles sold, how many sweaters of each style were sold?

1.3 Exercises

Find the solution of each of the following systems using Gauss elimination.

1. $\begin{aligned} x + 2y &= 12 \\ 2x + 3y &= 19 \end{aligned}$

2. $\begin{aligned} x + 3y &= 2 \\ 3x + 4y &= 1 \end{aligned}$

3. $\begin{aligned} 4x - 8y &= 20 \\ -x + 3y &= -7 \end{aligned}$

4. $\begin{aligned} 3x - 12y &= 3 \\ -2x + y &= -9 \end{aligned}$

5. $\begin{aligned} -2x + 8y &= -6 \\ -2x + 3y &= -1 \end{aligned}$

6. $\begin{aligned} -3x + 12y &= -21 \\ -10x + 2y &= 6 \end{aligned}$

7. $\begin{aligned} 3x + 6y &= 0 \\ x - y &= -3 \end{aligned}$

8. $\begin{aligned} 2x + 4y &= 8 \\ 2x - 4y &= 0 \end{aligned}$

9. $\begin{aligned} x + 2y &= 5 \\ 2x - 3y &= -4 \end{aligned}$

10. $\begin{aligned} 2x + 4y &= 6 \\ 4x - y &= -6 \end{aligned}$

11. $\begin{aligned} 3x - 3y + 6z &= -3 \\ 2x + y + 2z &= 4 \\ 2x - 2y + 5z &= -2 \end{aligned}$

12. $\begin{aligned} 2x - 4y + 2z &= -6 \\ 3x + 4y + 5z &= 1 \\ 2x - y + z &= -3 \end{aligned}$

13. $\begin{aligned} x + y + z &= 10 \\ x - y + z &= 10 \\ x + y - z &= 0 \end{aligned}$

14. $\begin{aligned} x + y + z &= 1 \\ 2x - y + z &= 2 \\ 3x + 2y + 5z &= 3 \end{aligned}$

15. $\begin{aligned} x + y + z &= 6 \\ 2x + y + 2z &= 10 \\ 3x + 2y + z &= 10 \end{aligned}$

16. $\begin{aligned} x + 2y + z &= 6 \\ -x + 3y - 2z &= -4 \\ 2x - y - 3z &= -8 \end{aligned}$

17. $\begin{aligned} x - y + 2z &= -1 \\ 2x + y - 3z &= 6 \\ y - z &= 2 \end{aligned}$

18. $\begin{aligned} x - 2y + z &= 9 \\ 3y - 2z &= -11 \\ x + y + 4z &= 3 \end{aligned}$

19. $\begin{aligned} 2x - 4y + 8z &= 2 \\ 2x + 3y + 2z &= 3 \\ 2x - 3y + 5z &= 0 \end{aligned}$

20. $\begin{aligned} 2x - y - 3z &= 1 \\ -x - 2y + z &= -4 \\ 3x + y - 2z &= 9 \end{aligned}$

21. $\begin{aligned} x + y + z + u &= 6 \\ y - z + 2u &= 4 \\ z + u &= 3 \\ x + 2y + 3z - u &= 5 \end{aligned}$

22. $\begin{aligned} x - y - z + 2u &= 1 \\ x - y + z + 3u &= 9 \\ y + 2z - u &= 5 \\ 3y + z + 2u &= 8 \end{aligned}$

23. $\begin{aligned} x + 2y + z - u &= -2 \\ x + 2y + 2z + 2u &= 9 \\ y + z - u &= -2 \\ y - 2z + 3u &= 4 \end{aligned}$

24. $\begin{aligned} x + y + z + u &= 6 \\ x + 2y - z + u &= 5 \\ x + y - z + 2u &= 6 \\ 2x + 2y + 2z + u &= 10 \end{aligned}$

25. A person has 25 coins, all of which are quarters and dimes. If the face value of the coins is \$3.25, how many of each type of coin must this person have?

26. A person has three times as many dimes as quarters. If the total face value of these coins is \$2.20, how many of each type of coin must this person have?

27. A person has 36 coins, all of which are nickels, dimes, and quarters. If there are twice as many dimes as nickels and if the face value of the coins is \$4, how many of each type of coin must this person have?

28. A person has three times as many nickels as quarters and three more dimes than nickels. If the total face value of these coins is \$2.40, how many of each type of coin must this person have?

Applications

29. Document Scheduling An insurance company has two types of documents to process: contracts and leases. The contracts need to be examined for 2 hours by the accountant and 3 hours by the attorney, while the leases needs to be examined for 4 hours by the accountant and 2 hours by the attorney. If the accountant has 40 hours and the attorney 28 hours each week to spend working on these documents, how many documents of each type can they process?

30. Tea Mixture A small store sells spearmint tea at \$3.20 an ounce and peppermint tea at \$4 an ounce. The store owner decides to make a batch of 100 ounces of tea that mixes both kinds and sell the mixture for \$3.50 an ounce. How many ounces of each of the two varieties of tea should be mixed to obtain the same revenue as selling them unmixed?

31. Investments at Two Banks An individual has a total of \$1000 in two banks. The first bank pays 8% a year and the second pays 10% a year. If the person receives \$86 of interest in the first year, how much money must have been deposited in each bank?

32. Meal Planning A dietitian must plan a meal for a patient using two fruits, oranges and strawberries. Each orange contains 1 gram of fiber and 75 mg of vitamin C, while each cup of strawberries contains 2 grams of fiber and 60 mg of vitamin C. How much of each of these fruits needs to be eaten so that a total of 8 grams of fiber and 420 mg of vitamin C will be obtained?

33. Production Scheduling for Shirts A small plant with a cutting department and a sewing department produce their Pathfinder shirt and their Trekking shirt. It takes 0.5 work-hours to cut the Pathfinder shirt and 0.6 work-hours to sew it. It takes 0.4 work-hours to cut the Trekker shirt and 0.3 work-hours to sew it. If the cutting department has 200 work-hours available each day and the sewing department has 186 work-hours, how many of each shirt can be produced per day if both departments work at full capacity?

34. Boutique Sales A boutique sells blouses and purses. The blouses cost \$30 each and the purses cost \$40 each. On a certain day the store sells twice as many blouses as purses. If the store sold \$400 worth of these two items on that day, how many of each were sold?

35. Document Scheduling An insurance company has three types of documents to process: appeals, claims, and enrollment forms. The appeal document needs to be examined for 2 hours by the accountant and 3 hours by the attorney. The claims document needs to be examined for 4 hours by the accountant and 2 hours by the attorney. Finally, the enrollment form document needs to be examined for 2 hours by the accountant and 4 hours by the attorney. The secretary needs 3 hours to type each document. If the accountant has 34 hours, the attorney 35 hours, and the secretary 36 hours available to spend working on these documents, how many documents of each type can they process?

36. Production Scheduling for Sweaters A small plant with a cutting department, a sewing department, and a packaging department produces three style of sweaters: crew, turtleneck, and V-neck. It takes 0.4 work-hours to cut a crew sweater and 0.2 work-hours to sew it. It takes 0.3 work-hours to cut a turtleneck sweater and the same to sew it. It takes 0.5 work-hours to cut a V-neck and 0.6 work-hours to sew it. It takes 0.1 work-hours to package each of the sweaters. If the cutting department has 110 work-hours available each day, the sewing department has 95 work-hours available each day, and the packaging department has 30 hours, how many sweaters of each styles can be produced if all departments work at full capacity?

37. Investments Jennifer has \$4200 to invest. She decides to invest in three different companies. The MathOne company costs \$20 per share and pays dividends of \$1 per share each year. The

NewModule company costs \$60 per share and pays dividends of \$2 per share each year. The JavaTime company costs \$20 per share and pays \$3 per share per year in dividends. Jennifer wants to have twice as much money in the MathOne company as in the JavaTime company. Jennifer also wants to earn \$290 in dividends per year. How much should Jennifer invest in each company to meet her goals?

*GC*38. **Investments** Link has \$14,800 to invest. He decides to invest in three different companies. The QX company costs \$40 per share and pays dividends of \$2 per share each year. The RY company costs \$120 per share and pays dividends of \$1 per share each year. The KZ company costs \$80 per share and pays \$2 per share per year in dividends. Link wants to have twice as much money in the RY company as in the KZ company. Link also wants to earn \$300 in dividends per year. How much should Link invest in each company to meet his goals?

39. **Production of Picture Frames** A company makes three kinds of picture frames. The frames use wood, paint, and glass. The number of units of each material needed for each type of frame is given in the table. How many of each type of frame can be made if there are 180 units of wood, 150 units of glass, and 130 units of paint available?

	Wood	Paint	Glass
Frame A	2	3	3
Frame B	2	2	2
Frame C	4	1	2

40. **Production of Furniture** A furniture company makes loungers, chairs, and footstools made out of wood, fabric, and stuffing. The number of units of each of these materials needed for each of the products is given in the table below. How many of each product can be made if there are 54 units of wood, 63 units of fabric, and 43 units of stuffing available?

	Wood	Fabric	Stuffing
Lounger	1	2	2
Chair	2	2	1
Footstool	3	1	1

41. **Baked Goods** A bakery sells muffins, scones, and croissants. Each batch of 6 muffins uses 2 cups of sugar and 3 cups of flour. Each batch of 10 scones uses 2 cups of sugar and 5 cups of flour. Each batch of 12 croissants uses 1 cup of sugar and 4 cups of flour. The bakery has 17 cups of sugar and 37 cups of flour. Muffins sell for \$2.50 each, scones sell for \$2.00 each, and croissants sell for \$1.50 each. How many of each item can be made if the revenue is \$169?

42. **Diet Planning** A dietitian is preparing a meal of chicken, rice, and peas for a patient. The patient needs the meal to contain 87 grams of carbohydrate, 57 grams of protein, and 7 grams of fat. The table below shows the number of grams of carbohydrate, protein, and fat in 100 grams of each food. (Note, the total is not 100 grams due to the water and fiber content of the food.) How much of each food should be used so that the patient gets the needed nutrients?

	Carbohydrate	Protein	Fat
Chicken	0	32	4
Rice	25	2	0
Peas	12	3	1

43. **Meal Planning** A dietitian wishes to design a meal for Sandy that will have her minimum daily requirements of iron, calcium, and folic acid. The dietitian will use Foods I, II, and III to make this meal. The table below shows how many units of each nutrient is found in each ounce of the foods. If Sandy needs 51 units of iron, 540 units of calcium, and 128 units of folic acid, how much of each food should she have in her meal?

	Iron	Calcium	Folic Acid
Food I	5	20	8
Food II	1	60	4
Food III	3	40	10

44. **Chinese Farm Problem** The ancient Chinese "way of calculating with arrays" can be found in Chapter 8 of the ancient text *Nine Chapters on Mathematical Art*. The following is the first problem listed in Chapter 8.

There are three grades of corn. After threshing, three bundles of top grade, two bundles of medium grade, and one bundle of low grade make 39 dou (a measure of volume). Two bundles of top grade, three bundles of medium grade, and one bundle of low grade make 34 dou. The yield of one bundle of top grade, two bundles of medium grade, and three bundles of low grade make 26 dou. How many dou are contained in each bundle of each grade?

45. Nutrition A dietitian must plan a meal for a patient using three fruits: oranges, strawberries, and blackberries. Each orange contains 1 gram of fiber, 75 mg of vitamin C, and 50 mg of phosphorus. Each cup of strawberries contains 2 grams of fiber, 60 mg of vitamin C, and 50 mg of phosphorus. Each cup of blackberries contains 6 grams of fiber, 30 mg of vitamin C, and 40 mg of phosphorus. How much of each of these fruits needs to be eaten so that a total of 13 grams of fiber, 375 mg of vitamin C, and 290 mg of phosphorus is obtained?

Extensions

46. Investments An individual wants to invest \$100,000 in four investment vehicles: a money market fund, a bond fund, a conservative stock fund, and a speculative stock fund. These four investments are assumed to have annual returns of 6%, 8%, 10%, and 13% respectively. The investor wants the same amount in the money market as in the speculative stock fund and wants the same amount in the bond fund as in the conservative stock fund. Can the investor yield \$8000 per year using the given restrictions? Why or why not?

47. Diet A dietitian will design a meal for Alice, who is in the hospital, that will have her minimum daily requirements of vitamin A, vitamin C, and copper. The dietitian wishes to use three foods readily available at the hospital: Food I, Food II, and Food III. The table below shows how many units of each nutrient is found in each ounce of the foods. Alice needs 37 units of vitamin A, 36 units of copper, and 177 units of vitamin C. Can the dietitian successfully find the amount of the three foods to satisfy Alice's needs?

	Vitamin A	Copper	Vitamin C
Food I	1	2	12
Food II	4	6	3
Food III	2	2	21

48. Biological system Suppose that in a biological system there are four species of animals with populations, x, y, z, and u, and four sources of food represented by the available daily supply. Let the system of linear equations be

$$\begin{aligned} x + 2y \phantom{{}+2z} + 3u &= 3500 \quad \text{Food 1} \\ x \phantom{{}+2y} + 2z + 2u &= 2700 \quad \text{Food 2} \\ z + u &= 900 \quad \text{Food 3} \\ 3z + 2u &= 3000 \quad \text{Food 4} \end{aligned}$$

Suppose $(x, y, z, u) = (1000, 500, 350, 400)$. Is there sufficient food to satisfy the average daily consumption?

49. Biological system Using the information in the question above, what is the maximum number of animals of the first species that could be individually added to the system with the supply of food still meeting the consumption?

50. Biological system Refer to the previous exercises. If the first species becomes extinct ($x = 0$), how much of an individual increase in the second species could be supported?

Solutions to Self-Help Exercises 1.3

1. Define the variables first. Let

$x =$ the number of style A sweaters sold,
$y =$ the number of style B sweaters sold,
$z =$ the number of style C sweaters sold.

Since a total of 30 were sold, we have $x+y+z=30$. The revenue in dollars from style A is $30x$, from style B is $40y$, and from style C is $50z$. Since \$1340 worth of sweaters were sold, we must have $30x+40y+50z=1340$. Finally, since the number of style C exceeded by 6 the sum of the other two styles, we also have $z=x+y+6$. Thus, we need to solve the system

$$\begin{aligned} x + \quad y + \quad z &= \quad 30 \text{ total sweaters} \\ 30x + 40y + 50z &= 1340 \text{ total revenue in \$} \\ -x - \quad y + \quad z &= \quad 6 \text{ ratio of sweaters} \end{aligned}$$

Writing the augmented matrix and proceeding with Gauss elimination gives

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 30 \\ 30 & 40 & 50 & 1340 \\ -1 & -1 & 1 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 30R_1 \to R_2 \\ R_3 + R_1 \to R_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 30 \\ 0 & 10 & 20 & 440 \\ 0 & 0 & 2 & 36 \end{array}\right] \begin{array}{l} \\ \frac{1}{10}R_2 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{rrr|r} 1 & 1 & 1 & 30 \\ 0 & 1 & 2 & 44 \\ 0 & 0 & 1 & 18 \end{array}\right]$$

This corresponds to the system of equations

$$\begin{aligned} x + y + \quad z &= 30 \\ y + 2z &= 44 \\ z &= 18 \end{aligned}$$

Backward substituting gives

$$\begin{aligned} E_2 : \qquad y + \quad 2z &= 44 \\ y + 2(18) &= 44 \\ y \qquad &= \ 8 \end{aligned}$$

$$\begin{aligned} E_1 : x + y + \quad z &= 30 \\ x + 8 + \quad 18 &= 30 \\ x \qquad &= \ 4 \end{aligned}$$

Thus, $(x, y, z) = (4, 8, 18)$, and the store sold 4 of style A, 8 of style B, and 18 of style C.

1.4 Systems of Linear Equations With Non-Unique Solutions

We now consider systems of linear equations that may have no solution or have infinitely many solutions. Nonetheless, we will see that Gauss elimination with backward substitution still provides the best and most efficient way to solve these systems.

✧ Systems of Equations With Two Variables

We begin by examining systems of linear equations with two variables. This will let us graph the equations and make connections to the geometrical meaning of non-unique solutions.

EXAMPLE 1 A System With No Solution Solve the following system of linear equations using Gauss elimination and discuss geometrically.

$$\begin{aligned} E_1 &: \quad x - 2y = 3 \\ E_2 &: -2x + 4y = 1 \end{aligned}$$

Technology Option
The augmented matrix can be simplified with the `rref` command on the TI calculators as detailed in the Technology Notes of the previous section.

Solution Now use Gauss elimination and the augmented matrix.

$$\left[\begin{array}{rr|r} 1 & -2 & 4 \\ -2 & 4 & -1 \end{array}\right] \quad R_2 + 2R_1 \to R_2 \qquad \longrightarrow \qquad \left[\begin{array}{rr|r} 1 & -2 & 3 \\ 0 & 0 & 7 \end{array}\right]$$

This last matrix corresponds to the system of equations

$$\begin{aligned} x - 2y &= 3 \\ 0 &= 7 \end{aligned}$$

The last equation is $0 = 7$. Since this is not true, there is no solution.

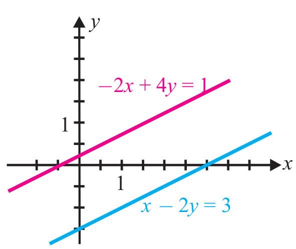


Figure 1.20

Let us now see what has happened geometrically. The first equation can be written as $y = 0.5x - 1.5$ and the second as $y = 0.5x + 0.25$. Both are graphed in Figure 1.20. We can readily see that the two lines are parallel. This verifies that there is no solution to the system of equations as two different parallel lines do not intersect. ✦

Next we examine systems of linear equations when the systems have infinitely many solutions.

EXAMPLE 2 Infinite Solutions Find all solutions to the following system of linear equations and discuss geometrically.

$$\begin{aligned} E_1 &: \quad x - 2y = \ \ 3 \\ E_2 &: -2x + 4y = -6 \end{aligned}$$

Solution Now use Gauss elimination and the augmented matrix.

$$\left[\begin{array}{rr|r} 1 & -2 & 3 \\ -2 & 4 & -6 \end{array}\right] \quad R_2 + 2R_1 \to R_2 \qquad \longrightarrow \qquad \left[\begin{array}{rr|r} 1 & -2 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

This last matrix corresponds to the system of equations

$$\begin{aligned} x - 2y &= 3 \\ 0 &= 0 \end{aligned}$$

The second equation $(0 = 0)$ is clearly true. We then see that there is only **one** equation yielding information and this indicates there are an infinitely number of solutions. Since we did not find a value for y from our second equation, we can let y be any number, say t. Substituting this into the first equation gives $x - 2t = 3$ or $x = 3 + 2t$ and the solutions can be written as $(x, y) = (3 + 2t, t)$. This type of solution is called a **parametric solution**. We call t the **parameter** and it can take on any real number as its value. We call $(x, y) = (3 + 2t, t)$ the **general solution** to this system of equations. Taking any particular value of t gives a **particular solution**. For example, taking $t = 0$, gives the particular solution $x = 3$, $y = 0$ and taking $t = 1$, gives the particular solution $x = 5$, $y = 1$.

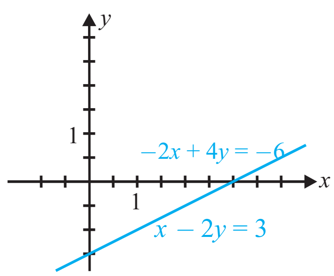


Figure 1.21

Let us observe this situation geometrically. Note that both equations can be written as $y = 0.5x - 1.5$ are they are graphed in Figure 1.21. This indeed implies that there are an infinite number of solutions to the system of equations because every point on one line is also on the other.

Larger Systems

We now consider larger systems of linear equations. We continue to use Gauss elimination to ensure that we find *all* the solutions and not just *some* of them.

EXAMPLE 3 **Infinite Number of Solutions** Solve the following system of linear equations.

$$\begin{aligned} E_1 &: \quad x + 2y + 3z = 4 \\ E_2 &: \quad 2x + 5y + 7z = 10 \\ E_3 &: \quad 2y + 2z = 4 \end{aligned}$$

Solution The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 10 \\ 0 & 2 & 2 & 4 \end{array}\right]$$

When the first nonzero element in a row is 1, we call it a **leading one** and circle it. We then have

$$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 3 & 4 \\ 2 & 5 & 7 & 10 \\ 0 & 2 & 2 & 4 \end{array}\right]$$

As before we use elementary row operations to obtain zeros in the first column below the circled leading one. Indeed, we will always want zeros below the circled leading one. Proceed with Gauss elimination to find

$$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 3 & 4 \\ 2 & 5 & 7 & 10 \\ 0 & 2 & 2 & 4 \end{array}\right] \quad R_2 - 2R_1 \rightarrow R_2 \quad \longrightarrow \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 3 & 4 \\ 0 & \textcircled{1} & 1 & 2 \\ 0 & 2 & 2 & 4 \end{array}\right] \quad R_3 - 2R_2 \rightarrow R_3$$

$$\longrightarrow \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 3 & 4 \\ 0 & \textcircled{1} & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

or

$$\begin{cases} E_1 : x + 2y + 3z = 4 \\ E_2 : \quad\quad y + \ \ z = 2 \\ E_3 : \quad\quad\quad\quad 0 = 0 \end{cases}$$

Technology Option
See Technology Note 1 on page 62 where Example 3 is solved using a graphing calculator

Notice that it is not possible to have a leading one in the last row since this row has all zeros.

We associate the **basic variables** with the columns with the leading ones. Thus, x and y are the basic variables. The remaining variables are called **free variables**. In this case, z is the free variable and we set z equal to the parameter t, $z = t$. Using $z = t$, we perform backward substitution to solve for the basic variables. We have

$$\begin{aligned} E_2: \quad\quad y + z &= 2 \\ y + t &= 2 \\ y &= 2 - t \end{aligned}$$

$$\begin{aligned} E_1: \quad\quad x + 2y + 3z &= 4 \\ x + 2(2 - t) + 3(t) &= 4 \\ x + 4 - 2t + 3t &= 4 \\ x &= -t \end{aligned}$$

Thus, the general solution can be written as $(x, y, z) = (-t, 2 - t, t)$, where t can be any real number. Some particular solutions to this system are (0, 2, 0), from using $t = 0$ and $(-1, 1, 1)$, from using the value $t = 1$. ✦

EXAMPLE 4 **Systems With an Infinite Number Of Solutions** Solve

$$\begin{aligned} E_1 &: \ \ x + 2y + 3z + \ \ 4u = \ \ 5 \\ E_2 &: 2x + 4y + 8z + 14u = 14 \\ E_3 &: \quad\quad\quad\quad 3z + 10u = \ \ 8 \end{aligned}$$

Solution We first form the augmented matrix, then proceed.

$$\left[\begin{array}{cccc|c} \textcircled{1} & 2 & 3 & 4 & 5 \\ 2 & 4 & 8 & 14 & 14 \\ 0 & 0 & 3 & 10 & 8 \end{array}\right] \quad R_2 - 2R_1 \rightarrow R_2 \quad \longrightarrow \left[\begin{array}{cccc|c} \textcircled{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 4 \\ 0 & 0 & 3 & 10 & 8 \end{array}\right]$$

Normally, our goal would be to get a 1 in the second row and second column. But we find a zero there. So we would normally switch the second row with a row below the second one. However, this will not help since the third row also has a zero in the second column. We then move over one element in the second row where we find a nonzero number. After dividing the second row by 2, this element will become our leading one. We do this and proceed as normal.

$$\left[\begin{array}{cccc|c} \textcircled{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 4 \\ 0 & 0 & 3 & 10 & 8 \end{array}\right] \quad \tfrac{1}{2}R_2 \to R_2 \quad \longrightarrow \quad \left[\begin{array}{cccc|c} \textcircled{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \textcircled{1} & 3 & 2 \\ 0 & 0 & 3 & 10 & 8 \end{array}\right] \quad R_3 - 3R_2 \to R_3$$

$$\longrightarrow \left[\begin{array}{cccc|c} \textcircled{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \textcircled{1} & 3 & 2 \\ 0 & 0 & 0 & \textcircled{1} & 2 \end{array}\right]$$

The variables associated with the columns with the leading ones are the basic variables.

$$\begin{array}{cccc} \textcircled{x} & y & \textcircled{z} & \textcircled{u} \end{array}$$

$$\left[\begin{array}{cccc|c} \textcircled{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \textcircled{1} & 3 & 2 \\ 0 & 0 & 0 & \textcircled{1} & 2 \end{array}\right]$$

Technology Option
See Technology Note 2 on page 62 to see Example 4 worked on a graphing calculator.

These are then x, z, and u, which have been circled. The remaining variable, y, is the free variable. We set $y = t$ and backward substitute.

$$E_3: \quad u = 2$$

$$\begin{aligned} E_2: \quad z + 3u &= 2 \\ z + 3(2) &= 2 \\ z &= -4 \end{aligned}$$

$$\begin{aligned} E_1: x + 2y + 3z + 4u &= 5 \\ x + 2(t) + 3(-4) + 4(2) &= 5 \\ x &= 9 - 2t \end{aligned}$$

The solution is $(x, y, z, u) = (9 - 2t, t, -4, 2)$, where t is any real number. ✦

Our goal is always to use elementary row operations to reduce the augmented matrix to a simple form where we can readily use backward substitution. When the augmented matrix is in this form, we call it the **echelon matrix**. More precisely, we have the following definition:

Echelon Matrix
A matrix is in **echelon form** if

1. The first nonzero element in any row is 1, called the leading one.
2. The column containing the leading one has all elements below the leading one equal to 0.
3. The leading one in any row is to the left of the leading one in a lower row.
4. Any row consisting of all zeros must be below any row with at least one nonzero element.

$$\begin{bmatrix} \textcircled{1} & x & x & x & x & x & x \\ 0 & \textcircled{1} & x & x & x & x & x \\ 0 & 0 & 0 & \textcircled{1} & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A typical matrix in echelon form is shown to the left. Notice the "staircase pattern." For example, all of the following matrices are in echelon form.

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 4 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 3 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

None of the following matrices are in echelon form

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 2 & 2 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 2 \end{array}\right], \quad \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 0 & 2 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

since the first matrix violates the first condition, the second matrix violates the second condition, the third matrix the third condition, and the fourth matrix the fourth condition.

REMARK: A matrix in echelon form may or may not have a solution.

EXAMPLE 5 **A System With Many Parameters** Find the solution to the the following system.

$$\begin{cases} E_1 : x + 2y + 3z + u + v = 4 \\ E_2 : \qquad\qquad z + 2u + 2v = 1 \\ E_3 : \qquad\qquad\qquad\qquad v = 3 \end{cases}$$

Solution The augmented matrix is

$$\begin{array}{c} \begin{array}{ccccc} x & y & z & u & v \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \end{array}$$

We see that this is already in echelon form. We circle the leading ones.

$$\begin{array}{c} \begin{array}{ccccc} \textcircled{x} & y & \textcircled{z} & u & \textcircled{v} \end{array} \\ \left[\begin{array}{ccccc|c} \textcircled{1} & 2 & 3 & 1 & 1 & 4 \\ 0 & 0 & \textcircled{1} & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 3 \end{array}\right] \end{array}$$

Technology Option
See Technology Note 3 on page 63 to see Example 5 worked on a graphing calculator.

We see that x, z, and v are basic variables. Thus, y and u are free variables. Set $y = s$ and $u = t$ and use backward substitution.

$$E_3 : \quad v = 3$$

$$E_2 : \quad \begin{aligned} z + 2u + 2v &= 1 \\ z + 2(t) + 2(3) &= 1 \\ z &= -5 - 2t \end{aligned}$$

$$E_1 : \quad \begin{aligned} x + 2y + 3z + u + v &= 4 \\ x + 2(s) + 3(-5 - 2t) + (t) + (3) &= 4 \\ x &= 16 - 2s + 5t \end{aligned}$$

So the solution is $(x, y, z, u, v) = (16 - 2s + 5t, s, -5 - 2t, t, 3)$ with s and t any real numbers. Particular solutions are found by choosing values for both s and t. ✦

✧ Geometric Interpretations

Earlier we looked at geometric interpretations in the case with two equations and two unknowns. We now consider higher dimensions. It can be shown (but we do not do so here) that any equation of the form $ax + by + cz = d$, with not all of the constants a, b, and c being zero, is the equation of a plane in space. If there are three equations with three unknowns, Figure 1.22 indicates some of the possibilities. The three planes could intersect at a single point indicating that the corresponding linear system of three equations in three unknowns has precisely one solution. The three planes could intersect in one line or an entire plane, giving an infinite number of solutions to the corresponding system. Or there could be no point of intersection of the three planes, indicating that the corresponding system has no solution.

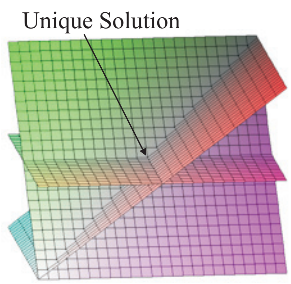


Figure 1.22a

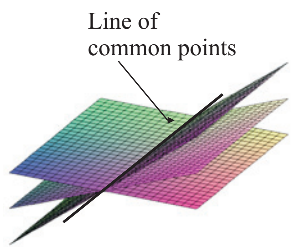


Figure 1.22b

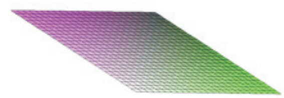


Figure 1.22c

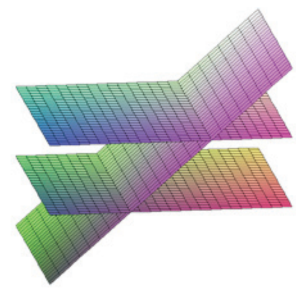


Figure 1.22d

The geometry can give us further insights. For example, from Figure 1.23 we can see that two linear equations in three unknowns cannot have precisely one solution. If two linear equations in three unknowns had precisely one solution, this would mean that the corresponding two planes intersected in precisely one point, but this is impossible. Thus, two linear equations in three unknowns has either no solution or an infinite number of solutions. This is true in general for any system with more variables than equations. A proof of this statement is outlined later under Enrichment (p. 59).

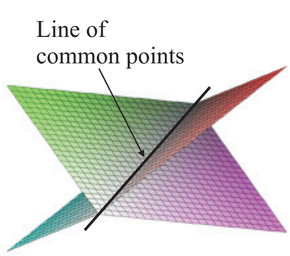


Figure 1.23a

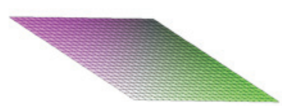


Figure 1.23b

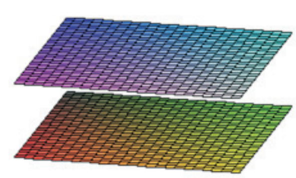


Figure 1.23c

✧ An Application

EXAMPLE 6 Purchasing Computers A firm must purchase a total of 100 computers, some of small, some of medium, and some of large capacity. The small capacity computers cost \$2000 each, the medium capacity computers cost \$6000 each, and the large capacity computers cost \$8000 each. If the firm plans to spend all of \$400,000 on the total purchase, find the number of each type to be bought.

Solution Let x, y, and z be the respective number of small, medium, and large capacity computers. Then the first sentence indicates that $x+y+z=100$.

The cost of purchasing x small computers is $2000x$ dollars, of purchasing y medium computers is $6000y$ dollars, and of purchasing z large computers is $8000z$ dollars. Since the total cost is \$400,000, we have $2000x+6000y+8000z=400,000$ or, in terms of thousands of dollars, $2x+6y+8z=400$. We then have the system

$$\begin{aligned} x + \ \ y + \ \ z &= 100 \\ 2x + 6y + 8z &= 400 \end{aligned}$$

Using the augmented matrix and using Gauss elimination, we obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 2 & 6 & 8 & 400 \end{array}\right] \quad R_2 - 2R_1 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 0 & 4 & 6 & 200 \end{array}\right] \quad \tfrac{1}{4}R_2 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 100 \\ 0 & 1 & 1.5 & 50 \end{array}\right]$$

This corresponds to

$$\begin{aligned} E_1 : x + y + \quad z &= 100 \\ E_2 : \quad y + 1.5z &= \ \ 50 \end{aligned}$$

The free variable is z and we let z be the parameter t. We have

$$\begin{aligned} E_2 : \quad & y + 1.5z = 50 \\ & y + 1.5t = 50 \\ & y = 50 - 1.5t \\ E_1 : \quad & x + y + z = 100 \\ & x + (50 - 1.5t) + (t) = 100 \\ & x = 0.5t + 50 \end{aligned}$$

This gives

$$\begin{aligned} x &= 0.5t + 50 \\ y &= 50 - 1.5t \\ z &= t \end{aligned}$$

First notice that since $t = z$ and z has units in terms of **integers**, t must be a nonnegative integer. We also have the following clues about the possible solutions to this system

- Both x and y must be integers and therefore t must be an *even* integer. So far, then, t must be a nonnegative even integer.
- Since x must be nonnegative, we must have $0.5t + 50 \geq 0$. This gives $t \geq -100$. However, this gives no new information.
- Since y must be nonnegative, we must have $50 - 1.5t \geq 0$. This gives $t \leq 100/3$. But recall that t must be a nonnegative *even* integer. So we must take t to be the largest even integer less than $100/3$, which is $t = 32$. Therefore, $t \leq 32$.

In summary, we have the solutions $(x, y, z) = (0.5t + 50, 50 - 1.5t, t)$, where t is an even integer with $0 \leq t \leq 32$. For example, the firm can take $t = 30$ giving $x = 0.5(30) + 50 = 65$ small capacity computers, $y = 50 - 1.5(30) = 35$ medium capacity computers, and $t = 30$ large capacity computers, spending the entire \$400,000 and obtaining a total of 100 computers. ✦

✧ Enrichment: A Proof

We mentioned earlier that when a system of equations has more variables than equations, then the system has no solution or an infinite number of solutions.

> **More Variables Than Equations**
> A system of linear equations that has more variables than equations has no solution or an infinite number of solutions.

By analyzing a typical reduced matrix, such as the one shown in Figure 1.25 we can see why this theorem must be true in general. Let us first assume the case in which there is no solution; that is, there is no row with all zeros except for the last entry. Then

$$\begin{aligned} \text{number of leading variables} &= \text{number of leading ones} \\ &\leq \text{number of equations} \\ &< \text{number of variables} \end{aligned}$$

Thus, there is one variable left to be free, and therefore, the system has an infinite number of solutions.

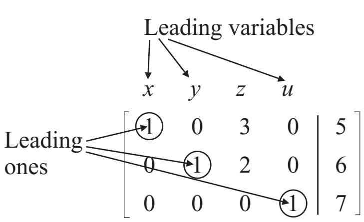


Figure 1.25

✧ Gauss-Jordan (Optional)

Recall in Example 4 that we ended with an echelon matrix that we then solved by backward substitution.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

It is possible, however, to continue to perform row operations to reduce this matrix to a simple form that allows us to read off the solution. Keep in mind, however, that the amount of calculations that we will do is no more nor less than what we did in backward substitution.

We start with the leading one in the last row and perform row operations to obtain zeros in the column above this leading one. We use the row with the lead one to aid in this in an efficient way. We then move to the next-to-the-last row with a leading one and perform row operations to obtain zeros in the column above this leading one. We use the row with this lead one to aid in an efficient way as shown below.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 4R_3 \to R_1 \\ R_2 - 3R_3 \to R_2 \end{array} \quad \to \left[\begin{array}{cccc|r} 1 & 2 & 3 & 0 & -3 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \; R_1 - 3R_2 \to R_1$$

$$\to \left[\begin{array}{cccc|r} 1 & 2 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

This can then be written as the following system.

$$\begin{aligned} x + 2y \qquad\quad &= \ \ 9 \\ z \quad &= -4 \\ u &= \ \ 2 \end{aligned}$$

As before, we have $u = 2$. But now we see that we immediately have $z = -4$. From our previous discussion, we know that y is the free variable. So we let $y = t$, where t is any real number. Now from the first equation, we have

$$\begin{aligned} x + \ \ 2y &= 9 \\ x + 2(t) &= 9 \\ x &= 9 - 2t \end{aligned}$$

The general solution is then $(x, y, z, u) = (9 - 2t, t, -4, 2)$ as before.

Consider a typical matrix that is already in echelon form as shown.

$$\left[\begin{array}{cccccc|c} \textcircled{1} & x & x & x & x & x & x \\ 0 & \textcircled{1} & x & x & x & x & x \\ 0 & 0 & 0 & \textcircled{1} & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We outline how the row operations would proceed.

$$\left[\begin{array}{cccccc|c} \textcircled{1} & x & x & x & x & x & x \\ 0 & \textcircled{1} & x & x & x & x & x \\ 0 & 0 & 0 & \textcircled{1} & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccccc|c} \textcircled{1} & x & x & x & x & x & 0 \\ 0 & \textcircled{1} & x & x & x & x & 0 \\ 0 & 0 & 0 & \textcircled{1} & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccccc|c} \textcircled{1} & x & x & 0 & x & x & 0 \\ 0 & \textcircled{1} & x & 0 & x & x & 0 \\ 0 & 0 & 0 & \textcircled{1} & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccccc|c} \textcircled{1} & 0 & x & 0 & x & x & 0 \\ 0 & \textcircled{1} & x & 0 & x & x & 0 \\ 0 & 0 & 0 & \textcircled{1} & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

✧ Enrichment: Efficient Calculations

We will now consider the issue of the best way of doing the precise sequence of row operations that will result in the least number of calculations. Consider for convenience, that we have a system of three linear equations with the variables u, v, and w, and do not need to use row swaps. We outline the procedure.

1. Make sure that the coefficient of u in the first equation is 1.

2. Use row operations with the first equation to eliminate u from the second and third equations.

3. Make sure that the coefficient of v in the second equation is 1.

4. Use a row operation with the second equation to eliminate w from the third equation.

5. Make sure that the coefficient of w in the third equation is 1. Stopping at this point gives

$$\left[\begin{array}{ccc|c} x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & x & x \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 1 & x \end{array}\right]$$

6. Now use row operations with the third equation to eliminate w from the first and second equations.

7. Use a row operation with the second equation to eliminate v from the first equation. Continuing we have

$$\left[\begin{array}{ccc|c} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 1 & x \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & x & 0 & x \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & x \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & x \end{array}\right]$$

In general, if you do row operations *other* than those outlined above, you will end up doing **more** calculations. You can actually prove this. Software created for solving systems always uses the procedure above to ensure that in general the process will use the least possible calculations. We therefore suggest that you follow this procedure in general so that you also can use the fewest possible calculations.

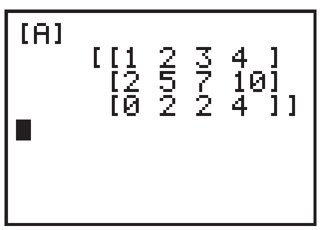
For example, doing the row operations in the following way will in general result in **more** calculations than that outlined above. See Exercises 66 and 67.

$$\left[\begin{array}{ccc|c} x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & x & x \\ 0 & 1 & x & x \\ 0 & 0 & x & x \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & x \end{array}\right]$$

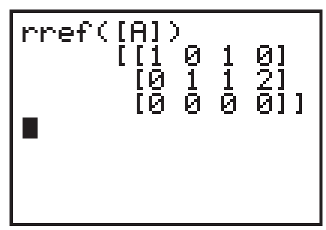
✧ Technology Corner

Technology Note 1 **Example 3 on a Graphing Calculator**

The calculator can do the Gauss elimination and backward substitution, but it cannot write the general solution. But, as before we can read the information off of the RREF form that is returned from the calculator. Screen 1.27 shows the initial augmented matrix and Screen 1.28 shows the final matrix.



Screen 1.27



Screen 1.28

Note that the system of equations is slightly different in appearance (though, of course, the solution will be the same in the end) when the calculator puts the system in RREF form:

$$\begin{aligned} x \quad + z &= 0 \\ y + z &= 2 \\ 0 &= 0 \end{aligned}$$

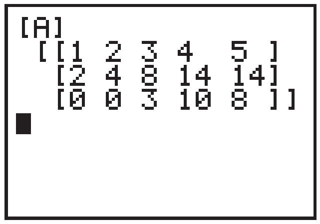
The basic variables are still those with the leading ones, x and y in this system and z will be the parameter. Substitute $z = t$ into the system and simplify,

$$\begin{aligned} x \quad + t = 0 &\rightarrow x = -t \\ y + t = 2 &\rightarrow y = 2 - t \end{aligned}$$

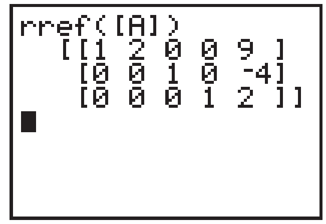
The solution, as before, is $(x, y, z) = (-t, 2 - t, t)$ where t can be any real number.

Technology Note 2 Example 4 on a Graphing Calculator

The initial and final matrices are shown in Screens 1.29 and 1.30.



Screen 1.29



Screen 1.30

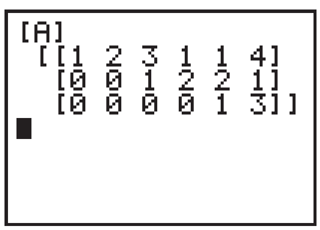
The system of equations in Screen 1.30 is

$$\begin{aligned} x + 2y \qquad &= \ \ 9 \\ z \quad &= -4 \\ u &= \ \ 2 \end{aligned}$$

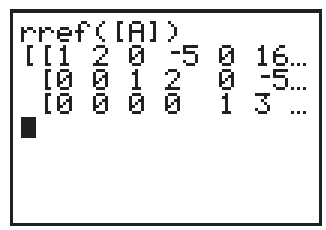
The basic variables are x, z, and u. Set the free variable $y = t$ as above and find $x = 9 - 2z$. The solution is $(x, y, z, u) = (9 - 2t, t, -4, 2)$, where t is any real number.

Technology Note 3 Example 5 on a Graphing Calculator

Using the calculator will give the same solution, as shown in Screens 1.31 and 1.32.



Screen 1.31



Screen 1.32

The system of equations from the RREF matrix is shown below. The columns without the leading ones have already been labeled as parameters. That is $y = s$ and $u = t$.

$$\begin{array}{ccccc} \textcircled{x} & s & \textcircled{z} & t & \textcircled{v} \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & -5 & 0 & 16 \\ 0 & 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \rightarrow \begin{aligned} x + 2s \quad - 5t \quad &= 16 \\ z + 2t \quad &= -5 \\ v \quad &= 3 \end{aligned}$$

$$\rightarrow \quad \begin{aligned} x &= 16 - 2s + 5t \\ z &= -5 - 2t \\ v &= 3 \end{aligned}$$

Putting this in the form $(x, y, z, u, v) = (16 - 2s + 5t, s, -5 - 2t, t, 3)$, with s and t any real number, we see the solution is the same.

Self-Help Exercises 1.4

1. Given the following augmented matrix, find the solution to the corresponding system.

$$\begin{array}{c} \begin{matrix} x & y & z & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

2. Given the following augmented matrix, find the solution to the corresponding system.

$$\begin{array}{c} \begin{matrix} x & y & z & \end{matrix} \\ \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

3. A contractor has 2000 hours of labor available for three projects: a fence, a deck, and a porch. The cost per work-hour for each of the three projects is \$10, \$12, and \$14, respectively, and the total labor cost is \$25,000. Find the number of work-hours that should be allocated to each project if all the available work-hours are to be used and all \$25,000 spent on labor.

1.4 Exercises

In Exercises 1 through 6 determine whether or not each matrix is in echelon form.

1. $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 1 & 3 & 2 \end{array}\right]$

2. $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

3. $\left[\begin{array}{ccccc|c} 1 & 2 & 4 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{array}\right]$

4. $\left[\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

5. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$

6. $\left[\begin{array}{ccccccc|c} 0 & 0 & 1 & 2 & 2 & 4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

In Exercises 7 through 16, each of the matrices is in echelon form. Write the corresponding linear system and solve.

7. $\begin{array}{c} \begin{matrix} x & y & \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right] \end{array}$

8. $\begin{array}{c} \begin{matrix} x & y & \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$

9. $\begin{array}{c} \begin{matrix} x & y & \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right] \end{array}$

10. $\begin{array}{c} \begin{matrix} x & y & z & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right] \end{array}$

11. $\begin{array}{c} \begin{matrix} x & y & z & u & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 5 \end{array}\right] \end{array}$

12. $\begin{array}{c} \begin{matrix} x & y & z & u & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{array}\right] \end{array}$

13. $\begin{array}{c} \begin{matrix} x & y & z & u & \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 6 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

14. $\begin{array}{c} \begin{matrix} x & y & z & \end{matrix} \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$

15.
$$\begin{array}{cccccc|c} x & y & z & u & v & w & \\ \end{array}$$
$$\left[\begin{array}{cccccc|c} 0 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

16.
$$\begin{array}{cccc} x & y & z & u \end{array}$$
$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

In Exercises 17 through 42, solve. You will find systems with an infinite number of solutions, with no solution, or with a unique solution.

17. $$\begin{aligned} 2x - 4y &= 8 \\ -x + 2y &= 4 \end{aligned}$$

18. $$\begin{aligned} -3x + 6y &= 12 \\ x - 2y &= 4 \end{aligned}$$

19. $$\begin{aligned} 3x - 6y &= 12 \\ -x + 2y &= -4 \end{aligned}$$

20. $$\begin{aligned} -2x + 4y &= 12 \\ x - 2y &= -6 \end{aligned}$$

21. $$\begin{aligned} -x + 3y &= 7 \\ 2x - 6y &= -14 \end{aligned}$$

22. $$\begin{aligned} x - 3y &= -8 \\ -2x + 6y &= 16 \end{aligned}$$

23. $$\begin{aligned} 0.1x - 0.3y &= 0.4 \\ -2x + 6y &= 4 \end{aligned}$$

24. $$\begin{aligned} -0.1x + 0.3y &= 0.5 \\ 2x - 6y &= 1 \end{aligned}$$

25. $$\begin{aligned} -x + 2y + 3z &= 14 \\ 2x - y + 2z &= 2 \end{aligned}$$

26. $$\begin{aligned} 2x - 2y + 2z &= -2 \\ 3x + y - z &= 5 \end{aligned}$$

27. $$\begin{aligned} -2x + 6y + 4z &= 12 \\ 3x - 9y - 6z &= -18 \end{aligned}$$

28. $$\begin{aligned} 3x - 3y + 6z &= 15 \\ -2x + 2y - 4z &= -10 \end{aligned}$$

29. $$\begin{aligned} -x + 5y + 3z &= 7 \\ 2x - 10y + 6z &= -14 \end{aligned}$$

30. $$\begin{aligned} x - y + 3z &= 4 \\ -2x + 2y - 6z &= -8 \end{aligned}$$

31. $$\begin{aligned} x + y + z &= 1 \\ x - y - z &= 2 \\ 3x + y + z &= 4 \end{aligned}$$

32. $$\begin{aligned} x - y - z &= 2 \\ 2x + y + z &= 1 \\ 4x - y - z &= 5 \end{aligned}$$

33. $$\begin{aligned} 2x + y - z &= 0 \\ 3x - y + 2z &= 1 \\ x - 2y + 3z &= 2 \end{aligned}$$

34. $$\begin{aligned} x - y + z &= 1 \\ 2x + 3y - 2z &= 1 \\ 3x + 2y - z &= 1 \end{aligned}$$

35. $$\begin{aligned} x - 2y + 2z &= 1 \\ 2x + y - z &= 2 \\ 3x - y + z &= 3 \end{aligned}$$

36. $$\begin{aligned} x + y - 2z &= 2 \\ -x + 2y + z &= 3 \\ x + 4y - 3z &= 7 \end{aligned}$$

37. $$\begin{aligned} x + y &= 4 \\ 2x - 3y &= -7 \\ 3x - 4y &= -9 \end{aligned}$$

38. $$\begin{aligned} x - 3y &= -7 \\ 3x + 4y &= 5 \\ 2x - y &= -4 \end{aligned}$$

39. $$\begin{aligned} x + 2y &= 4 \\ 2x - 3y &= 5 \\ x - 5y &= 2 \end{aligned}$$

40. $$\begin{aligned} x - 3y &= 2 \\ 3x + 2y &= 1 \\ 2x + 5y &= 1 \end{aligned}$$

41. $$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 2 \\ 3x + y + 3z &= 1 \end{aligned}$$

42. $$\begin{aligned} x + y + z &= 1 \\ x - y - z &= 1 \\ 4x + 4y + 2z &= 2 \end{aligned}$$

Perform the elementary row operations to find the solution to the systems in Exercises 43 and 44.

GC **43.** $$\begin{aligned} 1.2x + 2.1y + 3.4z &= 54.2 \\ 2.1x + 5.1y + 1.4z &= 72.7 \\ 3.7x + 1.4y + 3.8z &= 62.9 \end{aligned}$$

GC **44.** $$\begin{aligned} 2.3x + 1.04y + 3.65z &= 44.554 \\ 1.32x + 2.87y + 5.21z &= 59.923 \\ 1.66x + 1.92y + 3.41z &= 42.617 \end{aligned}$$

For Exercises 45 through 47 graph the equations to see if there is a solution.

45.
$$x + 2y = 4$$
$$2x - 3y = 5$$
$$x - 5y = 3$$

46.
$$x - 3y = 2$$
$$3x + 2y = 1$$
$$2x + 5y = 2$$

47.
$$x - 3y = 2$$
$$3x + 2y = 1$$
$$4x - y = 3$$

48. A person has three more dimes than nickels. If the total value of these coins is $2.40, how many of each type of coin must this person have?

49. A person has 36 coins, all of which are nickels, dimes, and quarters. If the total value of the coins is $4, how many of each type of coin must this person have?

Applications

50. Investments A pension fund manager has been given a total of $900,000 by a corporate client to invest in certain types of stocks and bonds over the next year. Stocks must be restricted to a certain class of low-risk stocks and another class of medium-risk stocks. Furthermore, the client demands an annual return of 8%. The pension fund manager assumes from historical data that the low-risk stocks should return 9% annually, the medium-risk stocks 11% annually, and bonds 7%. If the client demands an annual return of 9%, what should the pension fund manager do?

51. Cake Baking A bakery makes three kinds of special cakes. The Moon cake uses 2 cups of butter and 2 cups of sugar. The Spoon cake uses 1 cup of butter and 2 cups of sugar. The Loon cake uses 7 cups of butter and 4 cups of sugar. The bakery has 30 cups of butter and 30 cups of sugar available to make cakes today. How many of each type of cake can be made to use all the sugar and butter available?

52. Production Scheduling A small plant with a cutting department, a sewing department, and a packaging department produces three styles of sweaters: crew, turtleneck, and V-neck. It takes 0.4 work-hours to cut a crew sweater and 0.2 work-hours to sew it. It takes 0.3 work-hours to cut a turtleneck and the same to sew it. It takes 0.5 work-hours to cut a V-neck and 0.6 work-hours to sew it. If every day the cutting department has 110 work-hours available, the sewing department has 95 work-hours available, and the packaging department has 30 work-hours available, how many sweaters of each style can be produced eachday if all departments work at full capacity?

53. Sales of Sandals A store sells a total of 300 sandals in a certain month. The sandals come in three styles: Tiderunner, Sport, and Stone Harbor. The sales of the Sport sandal equaled the sum of the other two. The Tiderunner sandal costs $30, the Sport costs $40, and the Stone Harbor costs $50. If the store sold $11,500 worth of these sandals in that month, how many sandals of each style were sold?

54. Production Each day a firm produces 100 units of a perishable ingredient I and 200 units of another perishable ingredient II, both of which are used to manufacture four products, X, Y, Z, and U. The following table indicates how many units of each of the two perishable ingredients are used to manufacture each unit of the four products.

	I	II
X	0.1	0.4
Y	0.3	0.2
Z	0.6	0.4
U	0.2	0.3

Let x, y, z, and u be respectively the number of units of X, Y, Z, and U that are manufactured each day. If the manufacturer wishes to use the perishable ingredient that day, what are the options for the amounts of the four products to be manufactured?

55. Production Refer to the previous exercise. Just before production for the day is to begin, an order for 500 units of product U is received. The production manager is told to fill this order this very same day (while still using all the perishable ingredients) or be fired. Will the production manager be fired? Why or why not?

56. Production A woodworking company makes foot-stools, coffee tables, and rocking chairs.

Each spends time in a fabrication shop, a gluing shop, and a painting shop. The following table shows the number of hours each item spends in each shop. If there are 42 hours available in the fabrication shop, 30 hours in the gluing shop, and 24 hours in the painting shop, how many of each item can be made if all the available hours are to be used?

	Fabrication shop hours	Gluing shop hours	Painting shop hours
Foot − stool	1	0.5	1
Coffee table	2	1.5	1
Rocking chair	8	7	2

57. Scheduling Documents An insurance company has three types of documents to process: appeals, claims, and enrollment forms. The appeals document needs to be examined for 2 hours by the accountant and 3 hours by the attorney. The claims document needs to be examined for 4 hours by the accountant and 2 hours by the attorney. Finally, the enrollment forms document needs to be examined for 2 hours by the accountant and 4 hours by the attorney. If the accountant has 34 hours and the attorney 35 hours, how many documents of each type can they process?

Extensions

58. What condition must a, b, and c satisfy so that the following system has a solution?

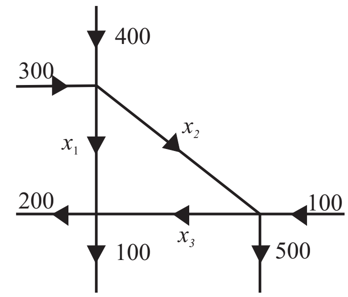
$$\begin{aligned} x + 2y &= a \\ 3x + 4y &= b \\ 2x + 3y &= c \end{aligned}$$

59. Show that the system of equations

$$\begin{aligned} x + 2y + az &= b \\ 2x + 5y + cz &= d \end{aligned}$$

has a solution no matter what a, b, c, and d are.

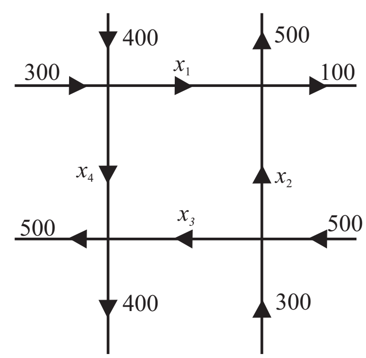
60. Traffic Flow The accompanying flow diagram indicates the traffic flow into and out of three intersections during rush hour. Traffic lights placed at each intersection can be timed to control the flow of traffic. Find all possible meaningful flow patterns.



61. Traffic Flow The accompanying flow diagram indicates the traffic flow in numbers of vehicles per hour into and out of four intersections during rush hour. Traffic lights placed at each intersection can be timed to control the flow of traffic. Find all possible meaningful flow patterns. *Hint*: For each intersection write an equation that states that the total flow into the intersection must equal the total flow out. For example, the diagram indicates that

$$x_1 + x_4 = 300 + 400 = 700$$

Now, using the diagram, find the other three equations. Solve.



62. Demand and Supply Cotterill and Haller estimated the costs and prices in the cereal-manufacturing industry and developed a mathematical model for demand curves for Shredded Wheat and Grape Nuts. They found that when the price p of Shredded Wheat went up, not only did the demand x for Shredded Wheat go down, but the demand for Grape Nuts went up. In the same way, when the price of q of Grape Nuts went up, not only did the demand y for Grape Nuts go down, but the demand for Shredded Wheat went up. They found the demand equations approximated by

$$x = 1.00 - 0.43p + 0.18q$$
$$y = 1.34 + 0.17p - 0.56q$$

Let the supply equations be given as

$$p = 1 + 0.5x + y$$
$$q = 1 + x + y$$

Find the point of equilibrium, that is, the point at which all four equations are satisfied.

Source: Cotterill and Haller 1997

63. Demand and Supply When the price p of pork goes up, not only does the demand x for pork go down, but the demand for beef goes up. In the same way, when the price of q of beef goes up, not only does the demand y for beef go down, but the demand for pork goes up. Let the demand equations be given by

$$x = 6 - 2p + q \qquad y = 16 + p - 3q$$

and let the supply equations be given as

$$p = 1 + 0.5x + y \qquad q = 1 + x + y$$

Find the point of equilibrium, that is, the point at which all four equations are satisfied.

64. Investments An individual wants to invest \$100,000 in four investment vehicles: a money market fund, a bond fund, a conservative stock fund, and a speculative stock fund. These four investments are assumed to have annual returns of 6%, 8%, 10%, and 13%, respectively. The investor wants the same amount in the money market as in the speculative stock fund and wants the same amount in the bond fund as the sum of the amounts in the two stock funds. Can the investor yield \$9,000 per year using the given restrictions? Why or why not?

65. Investments. Use the information found in the previous exercise, except that the investor wants a yield of \$10,000 per year. How can the investor do this?

66. Efficient Calculations Consider the following system of linear equations.

$$\begin{aligned} 2x + 4y + 4z &= 10 \\ 3x + 2y + z &= 7 \\ 2x - y - z &= 4 \end{aligned}$$

a. First solve this system carefully counting the number of multiplications and divisions needed by exactly following the method shown in the text.

b. Now solve this system carefully counting the number of multiplications and divisions needed by exactly following the alternative method shown in the text in the Enrichment subsection (p. 61).

c. Which method required the least number of multiplications plus divisions?

67. Efficient Calculations Consider the following system of linear equations.

$$\begin{aligned} x + 2y - 3z &= 4 \\ 2x - y + z &= 0 \\ 5x - 3y + 2z &= -2 \end{aligned}$$

a. First solve this system carefully counting the number of multiplications and divisions needed by exactly following the method shown in the text.

b. Now solve this system carefully counting the number of multiplications and divisions needed by exactly following the alternative method shown in the text in the Enrichment subsection.

c. Which method required the least number of multiplications plus divisions?

Solutions to Self-Help Exercises 1.4

1. We have

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 - R_2 \to R_2 \quad \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This can be written as

$$\begin{aligned} x + 2y + z &= 1 \\ z &= 2 \end{aligned}$$

The leading variables are x and z, while y is the free variable. Since y is taken to be arbitrary, we set $y = t$, where t is any real number. Then $x + 2y + z = 1$ becomes $x + (t) + (2) = 1$ or $x = -1 - 2t$. The general solution is then $(-1 - 2t, t, 2)$, where t is any real number.

2. The matrix is already in echelon form. The leading variables are y and z. The free variable is x. We have $z = 2$. Setting x equal to the parameter t yields $y + z = 1$ or $y = 1 - 2 = -1$. The general solution is then $(t, -1, 2)$.

3. Let x, y, and z be the amount of work-hours allocated to, respectively, the first, second, and third projects. Since the total number of work-hours is 2000, then $x+y+z = 2000$. The labor cost for the first project is \$10x, for the second is \$14z. Since the total labor funds is \$25,000, we then have $10x + 12y + 14z = 25,000$. These two equations are then written as

$$\begin{aligned} x + y + z &= \quad 2000 \text{ work hours} \\ 10x + 12y + 14z &= 25,000 \text{ labor hours} \end{aligned}$$

We then have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2000 \\ 10 & 12 & 14 & 25,000 \end{array}\right] \quad R_2 - 10R_1 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2000 \\ 0 & 2 & 4 & 5000 \end{array}\right] \quad 0.5R_2 \to R_2$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2000 \\ 0 & 1 & 2 & 2500 \end{array}\right]$$

Since the free variable is z, we set $z = t$ where t is any nonnegative number. (Work-hours are nonnegative.) Using backward substitution we have

$$\begin{aligned} y + 2(t) &= 2500 \\ y &= 2500 - 2t \end{aligned}$$

$$\begin{aligned} x + y + z &= 2000 \\ x + (2500 - 2t) + (t) &= 2000 \\ x &= t - 500 \end{aligned}$$

Then the general solution is $(t-500, 2500-2t, t)$. To obtain a reasonable answer we must also have $t - 500 \geq 0$ and $2500 - 2t \geq 0$. This requires $500 \leq t \leq 1250$. Thus, the contractor can allocate $t - 500$ work-hours to the first project, $2500-2t$ to the second, and t to the third. For example, taking $t = 800$, means the contractor can allocate 300 work-hours to the first project, 900 work-hours to the first project, and 800 to the third and use all of the available work-hours with all of the \$25,000 spent on labor.

1.5 Method of Least Squares

APPLICATION
Technology and Productivity

In 1995 Cohen published a study correlating corporate spending on communications and computers (as a percent of all spending on equipment) with annual productivity growth. He collected data on 11 companies for the period from 1985-1989. This data is found in the following table.

x	0.06	0.11	0.16	0.20	0.22	0.25	0.33	0.33	0.47	0.62	0.78
y	-1.0	4.5	-0.6	4.2	0.4	0.1	0.4	1.4	1.1	3.4	5.5

x is the spending on communications and computers as a percent of all spending on equipment, and y is the annual productivity growth as a percent.

What is the equation of a line that best approximates this data? What can you conclude about the relationship between spending on communications and computers and annual productivity growth? See Example 3 for the answer.
Source: Cohen 1995

The Method of Least Squares

Let x be the number of insulated mugs produced and sold, and let p be the price of each mug. Suppose x_1 mugs were sold at a price of p_1 and x_2 mugs were sold at a price of p_2. If we then *assume* that the demand equation is linear, then of course there is only *one* straight line through these two points, and we can easily calculate the equation $y = ax + b$ of this line.

What if we have more than two data points? Suppose, as shown in Table 1.3, that we have five points available. Here, p_i are the prices in dollars for insulated mugs, and x_i is the corresponding demand for these insulated mugs in thousands of mugs sold per day.

x_i	1	2	3	5	9
p_i	10	9	8	7	5

Table 1.3

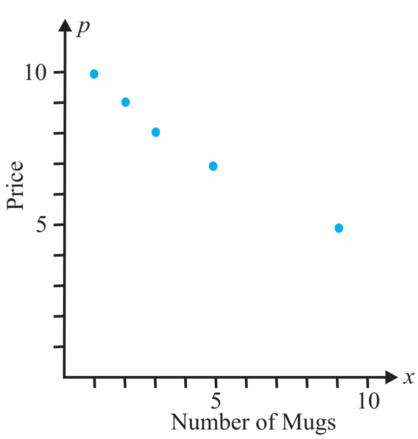


Figure 1.26

These are plotted in Figure 1.26, which is called a **scatter diagram**.

If we examine the scatter diagram, we see clearly that the points do not lie on any single straight line but seem to be scattered in a more or less linear fashion. Under such circumstances we might be justified in assuming that the demand equation was more or less a straight line. But what straight line? Any line that we draw will miss most of the points. We might then think to draw a line that is somehow closest to the data points. To actually follow such a procedure, we need to state exactly how we are to measure this closeness. We will measure this closeness in a manner that will lead us to the **method of least squares**.

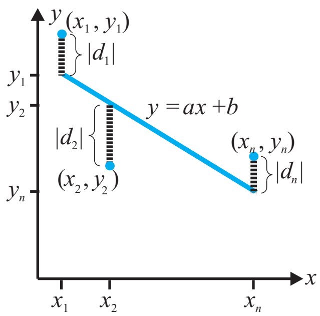


Figure 1.27

First notice that to be given a non-vertical straight line is the same as to be given two numbers a and b with the equation of the straight line given as $y = ax + b$. Suppose now we are given n data points (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) and a line $y = ax + b$. We then define $d_1 = y_1 - (ax_1 + b)$, and note from the figure that $|d_1|$ is just the vertical distance from the first data point (x_1, y_1) to the line $y = ax + b$. Doing the same for all the data points, we then have

$$
\begin{aligned}
d_1 &= y_1 - (ax_1 + b) \\
d_2 &= y_2 - (ax_2 + b) \\
&\vdots \\
d_n &= y_n - (ax_n + b)
\end{aligned}
$$

where $|d_2|$ is the vertical distance from the second data point (x_2, y_2) to the line $y = ax + b$, and so on. Refer to Figure 1.27.

Now if all the data points were on the line $y = ax + b$, then all the distances $|d_1|$, $|d_2|$, $\ldots$, $|d_n|$ would be zero. Unfortunately, this will rarely be the case. We then use the sum of the squares of these distances

$$d = d_1^2 + d_2^2 + \cdots + d_n^2$$

as a measure of how close the set of data points is to the line $y = ax + b$. Notice that this number d will be different for different straight lines: large if the straight line is far removed from the data points and small if the straight line passes close to all the data points. We then seek this line; that is, we need to find the two numbers a and b that will make this sum of squares the least. Thus the name **least squares**.

EXAMPLE 1 A Demand Function Find the best-fitting line through the data points in Table 1.3 and thus find a linear demand function. Then use the linear demand function to estimate the price if the demand is 6000 mugs. Finally, if the price is \$8.00 per mug, use the linear demand function to estimate the number of mugs that will be sold.

Solution Find the best-fitting line using the linear regression operation on a spreadsheet or graphing calculator. The steps are detailed in Technology Notes 1 and 2. You will find that $a = -0.6$ and $b = 10.2$. Thus, the equation of the best-fitting straight line that we are seeking is

$$y = p(x) = -0.6x + 10.2$$

The graph is shown in Figure 1.28. If technology is unavailable, the best-fitting line can be found by hand.

In general, the line $y = ax + b$ closest to the data points (x_1, y_1), (x_2, y_2), $\ldots$ (x_n, y_n) can be found by solving the following two linear equations for a and b:

$$
\begin{aligned}
(x_1^2 + \cdots + x_n^2)a + (x_1 + \cdots + x_n)b &= x_1y_1 + \cdots + x_ny_n \\
(x_1 + \cdots + x_n)a + \qquad\qquad nb &= y_1 + \cdots + y_n
\end{aligned}
$$

For our example we indicate these calculations by the following table.

HISTORICAL NOTE

Carl Gauss (1777–1855)

In 1781 Sir William Herschel discovered the planet Uranus, bringing the number of planets known at that time to seven. At that time astronomers were certain that an eighth planet existed between Mars and Jupiter, but the search for the missing planet had proved fruitless. On January 1, 1801, Giuseppe Piazzi, director of the Palermo Observatory in Italy, announced the discovery of a new planet Ceres in that location. (It was subsequently realized that Ceres was actually a large asteroid.) Unfortunately, he lost sight of it a few weeks later. Carl Gauss, one of the greatest mathematicians of all times, took up the challenge of determining its orbit from the few recorded observations. Gauss had in his possession one remarkable mathematical tool — the method of least squares — which he formulated in 1794 (at the age of 16!) but had not bothered to publish. Using this method, he predicted the orbit. Astronomers around the world were astonished to find Ceres exactly where Gauss said it would be. This incident catapulted him to fame.

	x_i	y_i	x_i^2	x_iy_i
	1	10	1	10
	2	9	4	18
	3	8	9	24
	5	7	25	35
	9	5	81	45
Sum	20	39	120	132

We then have the system of two equations in the two unknowns a and b.

$$\begin{aligned} 120a + 20b &= 132 \\ 20a + 5b &= 39 \end{aligned}$$

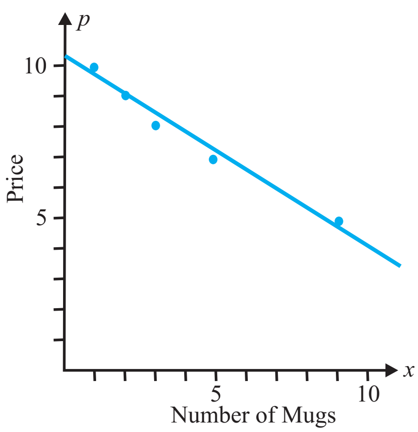


Figure 1.28

These equations can be readily solved using the techniques we used in the last several sections.

To find the price for 6000 mugs, we will use $x = 6$ in the demand equation.

$$p(6) = (-0.6)(6) + 10.2 = 6.6$$

That is, if 6000 mugs are to be sold, then the price should be \$6.60 each.

For a price of \$8.00 we set $p = 8$ and solve for x,

$$\begin{aligned} 8 &= -0.6x + 10.2 \\ 0.6x &= 2.2 \\ x &= \frac{2.2}{0.6} = \frac{11}{3} \approx 3.6667 \end{aligned}$$

Next note that since x represents thousands of mugs, we will expect to sell 3667 mugs at a price of \$8.00 each. ✦

✧ Correlation

We have just seen how to determine a functional relationship between two variables. This is called **regression analysis**. We now wish to determine the strength or degree of association between two variables. This is referred to as **correlation analysis**. The strength of association is measured by the **correlation coefficient** which is defined as

$$r = \frac{n\sum x_iy_i - \sum x_i \sum y_i}{\sqrt{[n\sum x_i^2 - (\sum x_i)^2][n\sum y_i^2 - (\sum y_i)^2]}}$$

where $\sum y_i$, for example, is $y_1 + y_2 + \cdots + y_n$. The value of this correlation coefficient ranges from $+1$ for two variables with perfect positive correlation to -1 for two variables with perfect negative correlation. See Figure 1.29 for examples.

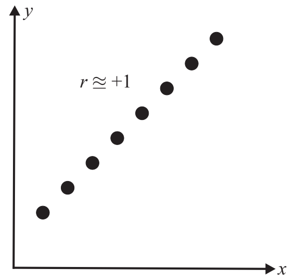


Figure 1.29a
Positive Correlation

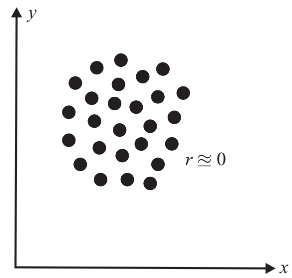


Figure 1.29a
No Correlation

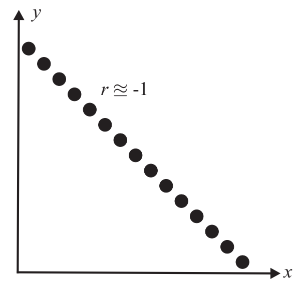


Figure 1.29a
Negative Correlation

Technology Option
Finding the correlation coefficient using a graphing calculator or Microsoft Excel is discussed in Technology Note 3 on page 76.

EXAMPLE 2 Correlation Find the correlation coefficient for the data in Table 1.3.

Solution We can use the above formula or let our computers or graphing calculators do the work. In doing Example 1 you will find that $r \approx -0.99$. This indicates a high negative correlation, which can easily be seen by observing the original data in Figure 1.26. We conclude that there is a strong negative correlation between price and demand in this instance. Thus, we expect increases in prices to lead to decreases in demand. ✦

✧ Additional Examples

There has been a debate recently as to whether corporate investment in computers has a positive effect on productivity. Some suggest that worker difficulties in adjusting to using computers might actually hinder productivity. The paper mentioned in the following example explores this question.

EXAMPLE 3 Technology and Productivity In 1995 Cohen published a study correlating corporate spending on communications and computers (as a percent of all spending on equipment) with annual productivity growth. He collected data on 11 companies for the period from 1985-1989. This data is found in the following table.

x	0.06	0.11	0.16	0.20	0.22	0.25	0.33	0.33	0.47	0.62	0.78
y	-1.0	4.5	-0.6	4.2	0.4	0.1	0.4	1.4	1.1	3.4	5.5

x is the spending on communications and computers as a percent of all spending on equipment, and y is the annual productivity growth as a percent.

Determine the best-fitting line using least squares and find the correlation coefficient. Discuss the results.

Source: Cohen 1995

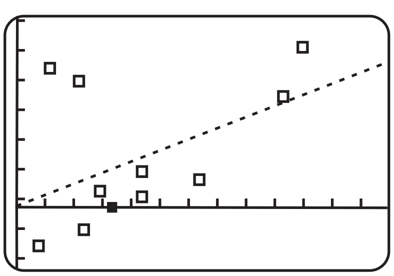


Figure 1.30

Solution Using a spreadsheet or a graphing calculator, we find that $a \approx 5.246$, $b \approx 0.080$, and $r \approx 0.518$. Thus, the best-fitting straight line is

$$y = ax + b = 5.246x + 0.080$$

The correlation coefficient is $r = 0.518$. This is somewhat significant and we say, on the basis of this study, that investment in communications and computers increases the productivity of corporations. See Figure 1.30 ✦

EXAMPLE 4 **Plant Costs** In 1997 Fuller and coworkers at Texas A&M University estimated the operating costs of cotton gin plants of various sizes. The operating costs of the smallest plant is shown in the following table.

x	2	4	6	8
y	109.40	187.24	273.36	367.60

x is annual number of thousands of bales produced, and y is the total cost in thousands of dollars

a. Determine the best-fitting line using least squares and the correlation coefficient.
b. The study noted that revenue was \$55.45 per bale. At what level of production will this plant break even?
c. What are the profits or losses when production is 1000 bales? 2000 bales?
Source: Fuller, Gillis, Parnell, Ramaiyer, and Childers 1997

Solution **a.** Using a spreadsheet or graphing calculator, we find that linear regression gives the cost equation as $y = C(x) = 43.036x + 19.22$ and $r = 0.9991$. This is certainly significant, as can be seen in Figure 1.31.

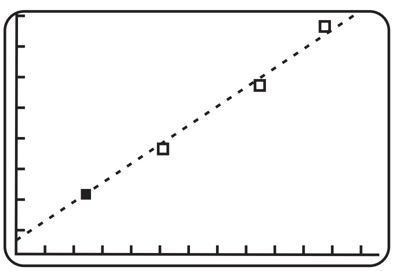


Figure 1.31

b. We set revenue equal to cost and obtain

$$\begin{aligned} R &= C \\ 55.45x &= 43.036x + 19.22 \\ 12.414x &= 19.22 \\ x &= 1.548 \end{aligned}$$

Thus, this plant will break even when production is set at approximately 1548 bales.

We can also solve this problem by finding profits, P, and set this equal to zero. We have profit is revenue less cost. Doing this we obtain

$$\begin{aligned} P &= R - C \\ &= 55.45x - (43.036x + 19.22) \\ &= 12.414x - 19.22 \end{aligned}$$

Setting $P = 0$, gives $12.414x = 19.22$ and $x = 1.548$ as before.

c. Since $P(x) = 12.414x - 19.22$, $P(1) = -6.806$ and $P(2) = 5.608$. Thus, there is a loss of \$6806 when production is at 1000 bales and a profit of \$5608 when production is set at 2000 bales. ✦

Technology Option
Example 4 is solved using a graphing calculator in Technology Note 4 on page 76

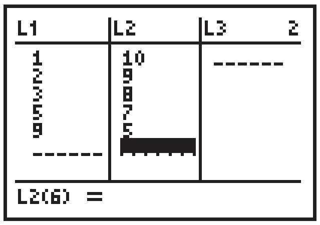
Technology Corner

Technology Note 1 **Example 1 on a Graphing Calculator**

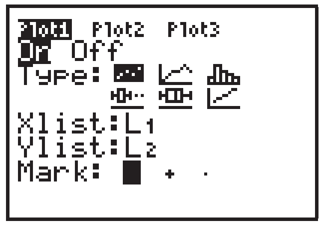
We can plot scatter diagrams on our calculator. To graph a scatter plot on your calculator, begin by entering the data into two lists as shown in Screen

1.33. The lists are accessed via the STAT button and choosing the 1:EDIT option. After all pairs of data have been entered, QUIT to the homescreen.

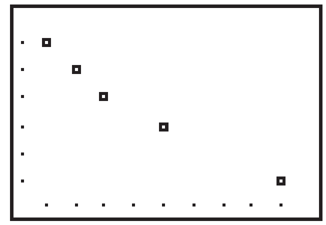
Set up the the STAT PLOT options by pressing 2ND and then Y= button. Press ENTER to edit the options for Plot1. The correct settings are shown in Screen 1.34. Note that the Xlist and Ylist names match the names of the lists that were edited. L1 and L2 are found above the keypad numbers 1 and 2 , respectively. The command 9:ZoomStat found under the ZOOM menu will find a window that fits the data to be displayed. The result is shown in Screen 1.35.



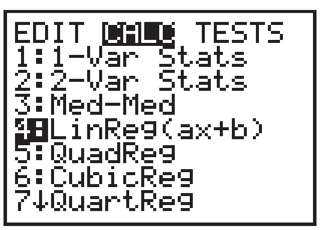
Screen 1.33



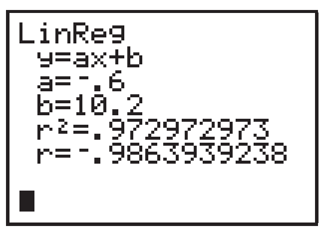
Screen 1.34


Screen 1.35

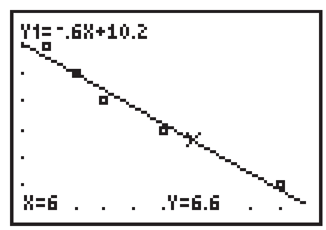
To find the best-fitting line, press STAT and right arrow to CALC and choose 4:LinReg(ax+b) as shown in Screen 1.36. Press ENTER to paste this on the homescreen. When LinReg(ax+b) is displayed on the homescreen, enter VARS , then arrow over to Y-VARS, then 1:Function and press ENTER to paste Y_1 next to the LinReg(ax+b) command. The results are displayed in Screen 1.37 and are also found under Y_1. See Technology Note 4 if your screen lacks the display of r^2 and r.



Screen 1.36



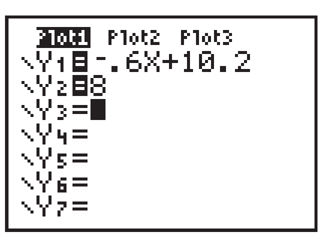
Screen 1.37



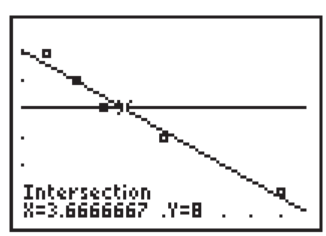
Screen 1.38

[.2,9.8]×[4.15,10.85]

To evaluate the regression equation with 6000 mugs, we need to evaluate the function at $x = 6$ (since it is in thousands of mugs). To do this, go to the CALC menu (above TRACE) and choose 1:value, as shown in Screen 1.39. You will be sent back to the graph and enter 6 for X and ENTER . The result is shown in Screen 1.40. To find the number of mugs sold at a price of \$8.00, enter 8 for Y2= and then GRAPH . Use CALC and 5:intersect to find the number of mugs in thousands.

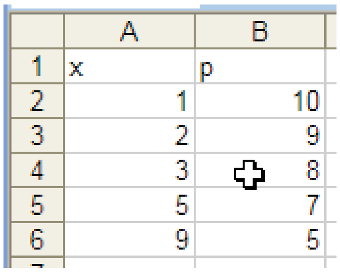


Screen 1.39



Screen 1.40

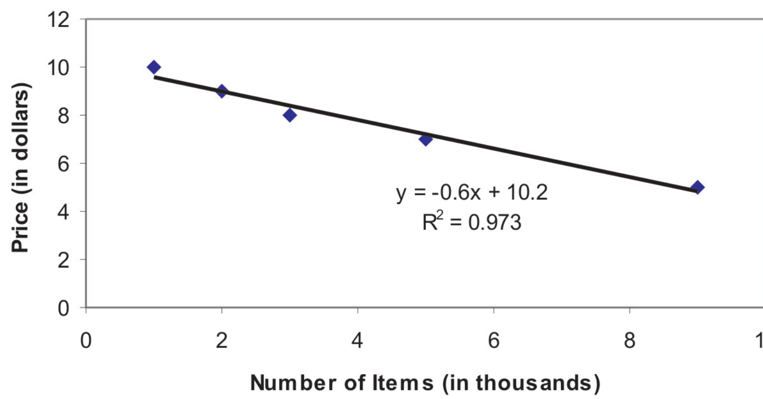
Technology Note 2 Example 1 with Microsoft Excel



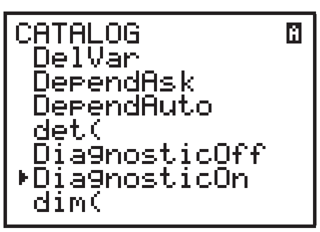
	A	B
1	x	p
2	1	10
3	2	9
4	3	8
5	5	7
6	9	5

Worksheet 1.1

Begin by opening a new worksheet. Label Column A with an x and Column B with a p. Enter the data from Table 1.3 as shown in Worksheet 1.1. Highlight the data and choose Chart Wizard from the tool bar or Insert and then Chart, if the Chart Wizard is not on the toolbar. Then pick the XY (Scatter) graph and choose the first chart type, Scatter. Compares pairs of values. Continue clicking Next until the chart is complete, adding titles and labels as needed. When the scatter plot is complete, click on the graph and then on Chart in the tool bar. Choose Add Trendline. . . . Pick the linear trend line and under the options menu, choose Display equation on chart and Display R-squared value on chart. The completed graph is shown below:



Technology Note 3 Correlation Coefficient

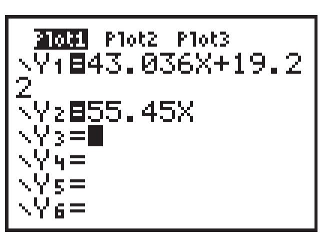


Screen 1.41

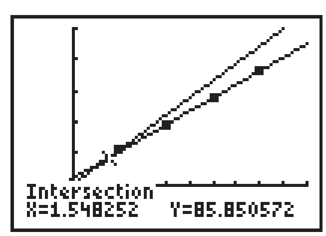
If your calculator does not display the correlation coefficient, r, it can be enabled by going to the CATALOG menu (above the 0) and scroll down to DiagnosticOn (see Screen 1.41) and press ENTER . Press ENTER again on the homescreen to enable the diagnostic. Next time a regression is done, the correlation coefficient, r, and the square of the correlation coefficient, r^2, will automatically be displayed. The correlation coefficient will be displayed on the Excel chart when this option is enabled. See Worksheet 1.2.

Technology Note 4 Example 4 on a Graphing Calculator

Enter the data into lists L1 and L2. Then, as in Technology Note 2, find the regression equation and enter it into Y_1= for our cost equation. In Y_2= enter the revenue equation, 55.45X, as shown in Screen 1.42. Screen 1.43.



Screen 1.42



Screen 1.43

[-2, 10] × [-100, 500]

Self-Help Exercises 1.5

1. Using the method of least squares, find the best-fitting line through the three points $(0, 0)$, $(2, 2)$, and $(3, 2)$. Find the correlation coefficient.

2. The table below shows x, the number of boxes of cereal in thousands that will be supplied at a price y, in dollars. Use the method of least squares and the supply information in the table to determine how many boxes of cereal will be supplied at a price of \$3.75.

x	9	7	6	4	3
y	4.49	3.05	2.49	2.10	1.92

1.5 Exercises

In Exercises 1 through 8, find the best-fitting straight line to the given set of data, using the method of least squares. Graph this straight line on a scatter diagram. Find the correlation coefficient.

1. $(0, 0)$, $(1, 2)$, $(2, 1)$

2. $(0, 1)$, $(1, 2)$, $(2, 2)$

3. $(0, 0)$, $(1, 1)$, $(2, 3)$, $(3, 3)$

4. $(0, 0)$, $(1, 2)$, $(2, 2)$, $(3, 0)$

5. $(1, 4)$, $(2, 2)$, $(3, 2)$, $(4, 1)$

6. $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 4)$

7. $(0, 4)$, $(1, 2)$, $(2, 2)$, $(3, 1)$, $(4, 1)$

Applications

8. Selling Strawberries The table below shows x, the number cartons of strawberries, that a fruit stand can sell at different prices y in dollars. Find the demand equation for strawberries using linear regression. Using the demand equation, find the price the stand should charge if they wish to sell 35 cartons of strawberries.

x	12	15	20	27	44	60
y	5.00	4.00	3.50	3.00	2.50	2.00

9. Selling Puzzles The table below shows x, the number of puzzles (in thousands), that the A-Mart company can sell at different prices, y in dollars. Find the demand equation for these puzzles using linear regression. Using that demand equation, find the price that A-Mart should charge if they wish to sell 10,000 puzzles.

x	8	5	12	3	15
y	5.00	6.00	3.50	8.00	3.00

10. Supply of Mugs The table below shows x, the number of mugs in thousands supplied by the Big Mug company for different prices in euros (y). Using linear regression find the supply equation and use it to determine the number of mugs that the Big Mug company will supply when the price is 3 euros.

x	3	5	6	13	18
y	1.50	1.80	2.00	3.20	4.75

11. Purchase Price of a Car The value of a 1998 car is given in the table below (value is in thousands of dollars). Find the depreciation equation using linear regression and use it to estimate to the nearest dollar the purchase price of this car.

Year	2000	2001	2002	2003	2004
Value	19	15	12	9	7

12. Purchase Price of a Machine The data in the table below is the value of a milling machine in thousands of dollars and the number of years since the item was purchased. Use linear regression to estimate the purchase price of this milling machine to the nearest dollar.

years since purchase	1	2	4	6	7
value in dollars	3.9	3.2	2.5	2	1.8

Referenced Applications

13. Deer Population and Deer-Vehicle Accidents Rondeau and Conrad studied the relationship of deer population in urban areas to accidents with vehicles on roads. Below is the data where x is the deer population in Irondequoit, NY, and y is the number of collisions with deer in the same city.

x	340	350	480	510	515	600	650
y	150	60	90	130	120	110	140

x	700	760	765	820	850
y	120	210	170	180	230

a. Determine the best-fitting line using least squares and the correlation coefficient.

b. Explain what the linear model is saying about the deer population and collisions.

Source: Rondeau and Conrad 2003

14. Economies of Scale in Plant Size *Strategic Management* relates a study in economies of scale in the machine-tool industry. The data is found in the following table,

x	70	115	130	190	195	400	450
y	1.1	1.0	0.85	0.75	0.85	0.67	0.50

where x is the plant capacity in thousands of units, and y is the employee-hours per unit.

a. Determine the best-fitting line using least squares and the correlation coefficient.

b. Is there an advantage in having a large plant? Explain.

c. What does this model predict the employee-hours per unit will be when the plant capacity is 300,000 units?

d. What does this model predict the plant capacity will be if the employee-hours per unit is 0.90?

Source: Rowe, Mason, Dickel, Mann, and Mockler 1994

15. Cost Curve Dean made a statistical estimation of the cost-output relationship in a hosiery mill. The data is given in the following table,

x	16	31	48	57	63	103	110
y	30	60	100	130	135	230	230

x	114	116	117	118	123	126
y	235	245	250	235	250	260

where x is the output in thousands of dozens and y is the total cost in thousands of dollars.

a. Determine the best-fitting line using least squares and the correlation coefficient. Graph.

b. What does this model predict the total cost will be when the output is 100,000 dozen?

c. What does this model predict the output will be if the total cost is $125,000?

Source: Dean 1976

16. Cost Curve Johnston estimated the cost-output relationship for 40 firms. The data for the fifth firm is given in the following table where x is the output in millions of units and y is the total cost in thousands of pounds sterling

x	180	210	215	230	260	290	340	400
y	130	180	205	190	215	220	250	300

x	405	430	430	450	470	490	510
y	285	305	325	330	330	340	375

a. Determine the best-fitting line using least squares and the correlation coefficient. Graph.

b. What does this model predict the total cost will be when the output is 300 million units?

c. What does this model predict the output will be if the total cost is 200,000 of pounds sterling?

Source: Johnston 1960

17. Productivity Bernstein studied the correlation between productivity growth and gross national product (GNP) growth of six countries. The countries were France (F), Germany (G), Italy (I), Japan (J), the United Kingdom (UK), and the United States (US). Productivity is given as output per employee-hour in manufacturing. The data they collected for the years 1950–1977 is given in the following table where x is the productivity growth (%) and y is the GNP growth (%).

	US	UK	F	I	G	J
x	2.5	2.7	5.2	5.6	5.7	9.0
y	3.5	2.3	4.9	4.9	5.7	8.5

a. Determine the best-fitting line using least squares and the correlation coefficient.

b. What does this model predict the GNP growth will be when the productivity growth is 7 percent?

c. What does this model predict the productivity growth will be if the GNP growth is 7%?

Source: Berstein 1980

18. Productivity Recall from Example 3 that Cohen studied the correlation between corporate spending on communications and computers (as a percent of all spending on equipment) and annual productivity growth. In Example 3 we looked at his data on 11 companies for the period from 1985 to 1989. The data found in the following table is for the years 1977–1984 where x is the spending on communications and computers as a percent of all spending on equipment and y is the annual productivity growth.

x	0.03	0.07	0.10	0.13	0.14	0.17
y	−2.0	−1.5	1.7	−0.6	2.2	0.3

x	0.24	0.29	0.39	0.62	0.83
y	1.3	4.2	3.4	4.0	−0.5

Determine the best-fitting line using least squares and the correlation coefficient.

Source: Cohen 1995

19. Environmental Entomology The fall armyworm has historically been a severe agricultural pest. How the age at first mating of the female of this pest affected its fecundity as measured by the number of viable larvae resulting from the eggs laid has been studied. The data is shown in following table, where x is the age of first mating of the female in days and y is the total number of viable larvae per female.

x	1	1	3	4	4	6	6
y	1650	1450	550	1150	650	850	800

x	8	10	10	12	13	15
y	450	900	500	100	100	200

Determine the best-fitting line using least squares. Also determine the correlation coefficient.

Source: Rogers and O. G. Marti 1994

20. Plant Resistance Talekar and Lin collected the data shown in the table that relates the pod diameter (seed size) of soybeans to the percentage of pods damaged by the limabean pod borer.

x	3.1	3.4	3.9	4.1	4.2	4.5
y	12	28	38	37	44	48

x is the pod diameter in mm and y is the percentage of damaged pods.

a. Use linear regression to find the best-fitting line that relates the pod diameter to the percentage of pods damaged.

b. Find the correlation coefficient.

c. Interpret what the slope of the line means.

Source: Talekar and Lin 1994

21. Horticultural Entomology The brown citrus aphid and the melon aphid both spread the citrus tristeza virus to fruit, and thus, have become important pests. Yokomi and coworkers collected the data found in the following table.

x	1	5	10	20
y	25	22	50	85
z	10	5	18	45

x is the number of aphids per plant, y and z are the percentage of times the virus is transmitted to the fruit for the brown and melon aphid, respectively.

a. Use linear regression for each aphid to find the best-fitting line that relates the number of aphids per plant to the percentage of times the virus is transmitted to the fruit.

b. Find the correlation coefficients.

c. Interpret what the slope of the line means in each case.

d. Which aphid is more destructive? Why?

Source: Yokomi, Lastra, Stoetzel, Damsteegt, Lee, Garnsey, Gottwald, Rocha-Pena, and Niblett 1994

22. Biological Control. Briano and colleagues studied the host-pathogen relationship between the black imported fire ant and a microsporidium (*T. solenopsae*) that infects them. This ant represents a serious medical and agricultural pest. The study was to determine whether *T. solenopae* could be used as a biological control of the imported fire ants. The table includes data that they collected and relates the number of colonies per hectare of the ants with the percentage that are infected with *T. Solenopsae*.

x	23	30	32	50	72	74	79	81	98	110
y	27	35	50	34	14	25	15	33	23	26

x	116	122	132	138	140	150	150	152	162
y	18	28	19	23	22	18	24	21	22

x is the colonies of ants per hectare, and y is the percentage of infected colonies.

a. Use linear regression to find the best-fitting line that relates the number of colonies of ants per hectare to the percentage of infected colonies.

b. Find the correlation coefficient.

c. Interpret what the slope of the line means.

Source: Briano, Patterson, and Cordo 1995

23. Federal Government Outlays The federal government budget outlays for recent years is given in the table.

x	2000	2001	2002	203	2004	2005
y	1.789	1.863	2.011	2.160	2.292	2.479

x is the year and y is budget outlays in billions of dollars.

a. Use linear regression to find the best-fitting line that relates the budget outlays in billions of dollars to the year.

b. Use this line to estimate the budget outlays for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

Source: Office of Management and Budget

24. Death Rate The table gives the death rate for the total population of the United States for recent years.

x	2000	2001	2002	2003	2004
y	8.7	8.5	8.5	8.3	8.1

x is the year and y is the death rate per 1000 total population in the United States.

a. Use linear regression to find the best-fitting line that relates the death rate to the year.

b. Use this line to estimated the death rate for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

Source: U.S. National Center for Health Statistics

25. Homeownership Rate The table gives the rate of homeownership in the United States for recent years.

x	1999	2000	2001	2002	2003
y	66.8	67.4	67.8	67.9	68.3

x is the year and y is the percent of family units who own houses.

a. Use linear regression to find the best-fitting line that relates the percent of homeownership to the year.

b. Use this line to estimate the homeownership rate for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

26. Corn Production. The table gives the U.S. production in millions of metric tons of corn.

x	1995	1999	2000	2004
y	188.0	240.0	251.9	300.0

x is the year and y is the U.S. production of corn in metric tons.

a. Use linear regression to find the best-fitting line that relates the U.S. production of corn to the year.

b. Use this line to estimate the U.S. production of corn for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

Source: U.S. Dept. of Agriculture

27. Gross Domestic Product The table gives the U.S. gross domestic product (GDP) in billions of dollars for recent years.

x	1990	2000	2003	2004
y	5803	9817	10,971	11,734

x is the year and y is the U.S. gross domestic product in billions of dollars.

a. Use linear regression to find the best-fitting line that relates the U.S. GDP to the year.

b. Use this line to estimate the U.S. GDB for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

Source: U.S. Bureau of Economic Analysis

28. Consumer Price Index The table gives the consumer price index(CPI) for recent years.

x	2000	2001	2002	2003	2004
y	172.2	177.1	179.9	184.0	188.9

x is the year and y is the CPI.

a. Use linear regression to find the best-fitting line that relates the U.S. CPI to the year.

b. Use this line to estimate the U.S. CPI for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

Source: U.S. Bureau of the Census

29. Consumer Credit The table gives the consumer credit outstanding in billions of dollars for some recent years.

x	2000	2002	2003	2004
y	1704	1923	2014	2105

x is the year and y is the consumer credit outstanding in billions of dollars.

a. Use linear regression to find the best-fitting line that relates the U.S. consumer credit outstanding to the year.

b. Use this line to estimate the U.S. consumer credit outstanding for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

Source: Board of Governors of the Federal Reserve System

30. Advertising The table shows the advertising expenditures in the U.S. (in billions of dollars) for recent years.

x	2000	2001	2002	2003	2004
y	244	231	236	245	264

x is the year and y the advertising expenditures in the U.S. in billions of dollars

a. Use linear regression to find the best-fitting line that relates the U.S. advertising expenditures to the year.

b. Use this line to estimate the U.S. advertising expenditures for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

Source: Advertising Age

31. World Crude Oil Production The table shows the world crude oil production (in million of barrels per day) in recent years.

x	1990	2000	2002	2003
y	60.57	68.34	66.84	69.32

x is the year and y the world crude oil production in million of barrels per day

a. Use linear regression to find the best-fitting line that relates the world crude oil production in millions of barrels per day to the year.

b. Use this line to estimate the world crude oil production in millions of barrels per day for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

Source: U.S. Dept. of Energy

32. Personal Computer Sales Average Price The table shows the average price of a personal computer in dollars for each year.

x	1999	2000	2001	2002
y	1100	1000	900	855

x is the year and y the average price of a personal computer.

a. Use linear regression to find the best-fitting line that relates the average price of a personal computer to the year.

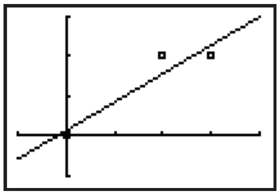
b. Use this line to estimate the price of a personal computer for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

Source: Consumer Electronics Association

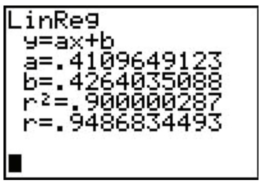
Solutions to Self-Help Exercises 1.5



1. Input the data into a spreadsheet or graphing calculator. Using the linear regression operation, we find that $a \approx 0.714$, $b \approx 0.143$, and $r \approx 0.9449$. Thus, the best-fitting straight line is

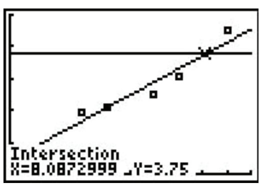
$$y = ax + b = 0.714x + 0.143$$

The correlation coefficient is $r = 0.9449$. This is significant. See the screen on the left.



2. Input the data into a spreadsheet or graphing calculator. Using the linear regression operation we find that $a \approx 0.41096$ and $b \approx 0.42640$. The correlation is quite significant at $r \approx 0.94868$, see the screen below. To find the number of boxes of cereal demanded at a price of \$3.75, either solve the equation

$$3.75 = 0.41096x + 0.42640 \rightarrow x \approx 8.0873$$



Or graph the regression equation and the line $p = 3.75$ and find the intersection, as shown in the screen on the left. The value $x \approx 8.0873$ means that 8087 boxes of cereal will be demanded.

1.R Review

✧ Summary Outline

- A **function** f from the set D to the set R is a rule that assigns to each element x in D one and only one element $y = f(x)$ in R.
- The set D above is called the **domain**.
- One thinks of the domain as the set of inputs and the values $y = f(x)$ as the outputs.
- The set of all possible outputs is called the **range**.
- The letter representing the elements in the domain is called the **independent variable**.
- The letter representing the elements in the range is called the **dependent variable**.
- The **graph** of the function f consists of all points (x, y) such that x is in the domain of f and $y = f(x)$.
- **Linear Cost, Revenue, and Profit Equations.** Let x be the number of items made and sold.

  variable cost = (cost per item) × (number of items produced) = mx.

  C = **cost** = (variable cost) + (fixed cost) = $mx + b$.

  R = **revenue** = (price per item) × (number sold) = px.

  P = **profit** = (revenue) − (cost) = $R - C$.
- The quantity at which the profit is zero is called the **break-even quantity**.
- Let x be the number of items made and sold and p the price of each item. A **linear demand equation**, which governs the behavior of the consumer, is of the form $p = mx + b$, where m must be negative. A **linear supply equation**, which governs the behavior of the producer, is of the form $p = mx + b$, where m must be positive.
- The point at which supply equals demand is called the **equilibrium point**. The x-coordinate of the equilibrium point is called the **equilibrium quantity**, and the p-coordinate is called the **equilibrium price**.
- A **system of linear equations** is a set of linear equations.
- The set of all solutions to a system of linear equations is called the **solution set**.
- Two systems of equations with the same solution set are said to be **equivalent**.
- A **matrix** is a rectangular array of numbers.

- There are three **elementary operations** performed on a system of equations (matrix). Interchange two equations (rows). Replace an equation (row) with a constant times the equation (row). Replace an equation (row) plus a constant times another equation (row).

- Elementary operations yield an equivalent system of equations.

- **Gauss** elimination is a procedure for a systematically using elementary operations to reduce a system of linear equations to an equivalent system whose solution set can be immediately determined by backward substitution.

- There are three possibilities in solving a system of linear equations: There can be precisely one solution. There can be no solution. There can be infinitely many solutions.

- A matrix is in **echelon form** if:
  1. The first nonzero element in any row is 1, called the leading one.
  2. The column containing the leading one has all elements below the leading one equal to 0.
  3. The leading one in any row is to the left of the leading one in a lower row.
  4. Any row consisting of all zeros must be below any row with at least one nonzero element.

- The leftmost nonzero element in a nonzero row of a matrix in echelon form is called a **leading one**.

- The variable in the column of a matrix in echelon form containing the leading one is called a **leading variable**. The remaining variables are called **free variables**.

- A **parameter** is any real number. Free variables are set equal to a parameter and are thus free to be any number.

1.R Exercises

1. Assuming a linear cost model, find the equation for the cost C, where x is the number produced, the cost per item is \$6, and the fixed cost is \$2000.

2. Assuming a linear revenue equation, find the revenue equation for R, where x is the number sold and the price per item is \$10.

3. Assuming the cost and revenue equations in the previous two exercises, find the profit equations. Also find the break-even quantity.

4. Given that the cost equation is $C = 5x + 3000$ and the revenue equation is $R = 25x$, find the break-even quantity.

5. Given the demand curve $p = -2x + 4000$ and the supply curve $p = x + 1000$, find the equilibrium point.

6. Demand for Pens A company notes from experience that it can sell 100,000 pens at \$1 each and 120,000 of the same pens at \$0.90 each. Find the demand equation, assuming it is linear.

7. Profit from Magazines It costs a publisher \$2 to produce a copy of a weekly magazine. The magazine sells for \$2.50 a copy, and the publisher obtains advertising revenue equal to 30% of the revenue from sales. How many copies must be sold to obtain a profit of \$15,000?

8. Oil Wildcatter An oil wildcatter has drilled an oil well at a cost of \$70,000. We will assume there are no variable costs. The revenue for this well is \$12,000 per year. Determine a profit equation $P(t)$, where t is given in years.

9. Oil Secondary Recovery After drilling, striking oil, and extracting an optimal amount of oil, a wildcatter decides to introduce secondary recovery techniques at a fixed cost of \$20,000. The revenue using this recovery technique should yield an additional \$2000 per year. Write a profit function $P(t)$ as a function of the years t, assuming no variable costs.

10. Rent or Buy Decision Analysis A forester wants to split many logs for sale for firewood. On the one hand he could, when needed, rent a log splitter for \$150 a day. Since he has a large amount of work to do, he is considering purchasing a log splitter for \$1400. He estimates that he will need to spend \$10 per day on maintenance.

a. Let x be the number of days he will use a log splitter. Write a formula that gives him the total cost of renting for x days.

b. Write a formula that gives him the total cost of buying and maintaining the log splitter for x days of use.

c. If the forester estimates he will need to use the splitter for 12 days, should he buy or rent?

d. Determine the number of days of use before the forester can save as much money by buying the splitter as opposed to renting.

In Exercises 11 through 18, find all the solutions, if any exist, of the given system.

11. $\begin{aligned} x + 3y &= 7 \\ 3x + 4y &= 11 \end{aligned}$

12. $\begin{aligned} x + 3y &= 1 \\ 3x + 9y &= 2 \end{aligned}$

13. $\begin{aligned} x + 3y &= 1 \\ 3x + 9y &= 3 \end{aligned}$

14. $\begin{aligned} 2x + 3y &= 18 \\ 3x - y &= 5 \end{aligned}$

15. $\begin{aligned} x + 3y - z &= 4 \\ 3x - y + z &= 4 \end{aligned}$

16. $\begin{aligned} x + y &= 10 \\ 2x + 2y + 2z &= 20 \end{aligned}$

17. $\begin{aligned} x + 2y - 3z &= 4 \\ 2x - y + z &= 0 \\ 5x - 3y + 2z &= -2 \end{aligned}$

18. $\begin{aligned} x + y + z &= 5 \\ 2x + y - z &= 3 \\ y + 3z &= 7 \end{aligned}$

19. Golf Course A certain 9-hole golf course has the number of par 3 holes equal to one plus the number of par 5 holes. If par at this course for 9 holes is 35, how many par 3, 4, and 5 holes are there?

20. Sales of Sweepers A wholesaler receives an order for a total of a dozen standard and deluxe electric sweepers. The standard sweepers cost \$200 and the deluxe \$300. With the order is a check for \$2900, but the order neglects to specify the number of each type of sweeper. Determine how to fill the order.

21. Sales of Condominiums A developer sells two sizes of condominiums. One sells for \$75,000 and the other \$100,000. One year the developer sold 14 condominiums for a total of \$12 million. How many of each size of condominium did the developer sell?

22. Sales of Specials A small restaurant sold 50 specials one day. The ham special went for \$10 and the beef special for \$12. The revenue from these specials is \$560. The cook was too busy to keep track of how many of each were served. Find the answer for her.

23. Acid Mixture Three acid solutions are available. The first has 20% acid, the second 30%, and the third 40%. The three solutions need to be mixed to form 300 liters of a solution with 32% acid. If the amount of the third solution must be equal to the sum of the amounts of the first two, find the amount of each solution.

24. Distribution A firm has three manufacturing plants located in different parts of the country. Three major wholesalers obtain the percentages of the product from the three plants according to the following table. Also included is the demand (in numbers) from each of the wholesalers. What should be the number produced by each of the plants to meet this demand with the given percentage distributions?

	Plant 1	Plant 2	Plant 3	Demand
Wholesaler A	20%	30%	50%	105
Wholesaler B	40%	20%	40%	100
Wholesaler C	10%	30%	60%	100

25. Using the method of least squares, find the best-fitting line through the four points $(0,0)$, $(2,2)$, $(3,2)$, and $(4,3)$.

26. Percent of U.S. Homes with Home Computer The table gives the percentage of U.S. homes with a personal computer for recent years.

x	1999	2000	2001	2002
y	48	54	58	60

x is the year and y the percentage of U.S. homes with a personal computer.

a. Use linear regression to find the best-fitting line that relates the percentage of U.S. homes with a personal computer to the year.

b. Use this line to estimate the percentage of U.S. homes with a personal computer for the year 2006.

c. Find the correlation coefficient.

d. Interpret what the slope of the line means.

Source: Telecommunications Association

✧ Project: Supply and Demand Dynamics

Let us consider some commodity, corn, to be specific. For this discussion we will find it convenient to have demand as a function of price. Thus, the price p will be on the horizontal axis, and the demand x will be on the vertical axis. Let us now suppose that the supply equation is $x = p - 1$ and the demand equation is $x = 11 - 2p$, where x is bushels and p is in dollars per bushel.

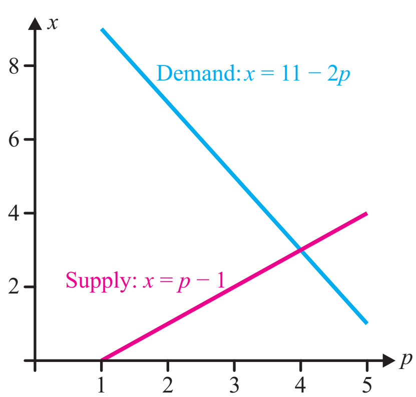


Figure 1.32
Supply and Demand for Corn

The graphs are shown in Figure 1.32. Notice from the supply curve that as the price decreases, the supply does also until a point is reached at which there will be no supply. Normally, there is a price at which suppliers refuse to produce anything (sometimes called the "choke" price). We can readily find the equilibrium price for our model:

$$
\begin{aligned}
p - 1 &= 11 - 2p \\
3p &= 12 \\
p &= 4
\end{aligned}
$$

We noted that in equilibrium, demand must equal supply, which corresponds to the point where the demand curve and the supply curve intersect. This rarely happens. What actually happens is the farmer bases production on the price p_n of corn that prevails at planting time. The supply equation will then determine the supply of corn. Owing to the time lag in the growing process, the resulting supply of corn is not available until the next period, that is, the fall. At that time the price will be determined by the demand equation. The price will adjust so that all the corn is sold. The farmer notes the new price p_{n+1} at planting time, and a new cycle begins.

The supply s_{n+1} is determined by the equation $s_{n+1} = p_n - 1$, and the demand is determined by the equation $d_{n+1} = 11 - 2p_{n+1}$ according to the discussion in the previous paragraph. We impose the realistic condition that the price (at the time the corn is brought to market) is adjusted so that demand equals supply, that is, so that all the corn is sold. Imposing the condition that supply must equal demand gives the equation

$$p_n - 1 = s_{n+1} = d_{n+1} = 11 - 2p_{n+1}$$

This gives

$$p_{n+1} = 6 - 0.5p_n$$

That is, the price next year, p_{n+1}, will be 6 less one-half times this year's price.

Now let's suppose that the price starts out at $p_0 = 2$. Then rounding off to the nearest cent we have

$$p_1 = 6 - 0.5(2) = 5$$

$$p_2 = 6 - 0.5(5) = 3.5$$

$$p_3 = 6 - 0.5(3.5) = 4.25$$

$$p_4 = 6 - 0.5(4.25) = 3.88$$

$$p_5 = 6 - 0.5(3.88) = 4.06$$

$$p_6 = 6 - 0.5(4.06) = 3.97$$

$$p_7 = 6 - 0.5(3.97) = 4.02$$

$$p_8 = 6 - 0.5(4.02) = 3.99$$

$$p_9 = 6 - 0.5(3.99) = 4.01$$

$$p_{10} = 6 - 0.5(4.01) = 4.00$$

So the price in this case moves steadily toward the equilibrium price.

There is a dramatic graphical way of determining and seeing the price movements. This graphical technique is called a **cobweb**. The cobweb is shown in Figure 1.33. We start at the point labeled A. The price is \$2. With such a low price, far below the equilibrium, farmers plant a small crop and bring the crop to market in the fall of that year. Fortunately for the farmer, the small supply results from the demand equation with a high price of \$5.00 (point B). This is obtained by drawing the horizontal line seen in Figure 1.33. Now proceed to the spring of the next year by drawing a vertical line until we hit the supply curve at point C. Now the farmer sees a high price of \$5 for corn, so the supply curve determines the supply to be produced, which is relatively large. When the corn is brought to market in the fall (follow the horizontal to the demand curve), the price is set by the demand curve at point D. The large amount of corn results in a poor price (for the farmer) of \$3.50. So the farmer produces little corn the following year, resulting in a higher price of \$4.25, point E, and so on. As the cobweb indicates, the price gets closer and closer to the equilibrium price.

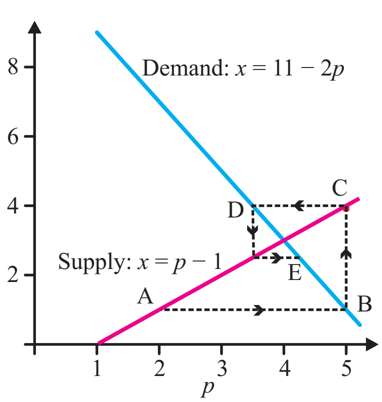


Figure 1.33

1.S Answers to Selected Exercises

EXERCISE SET 1.1

1. Cost: $C = 3x + 10,000$
3. Revenue: $R = 5x$
5. Profit: $P = 2x - 10,000$
7. $p = -0.4x + 14$
9. $p = -0.25x + 60$
11. $p = 1.6x + 15$
13. $p = 5x + 10$
15. $V = -15t + 135$
17. $V = -1500t + 15,000$
19. $C = 10x + 150$, x in hours
21. $C = 2x + 91,000$, x number of shirts
23. $C = 5x + 1000$
25. $p = -0.0005x + 1.5$
27. $R = 12x$
29. $C = -x + 2000$
31. $R = 11x$
33. $p = -0.05x + 4.53$
35. $p = -0.05 + 1.25$
37. $V = -5000t + 50,000$
    After 1 year, $V(1) = 45,000$
    After 5 year, $V(5) = 25,000$
39. Fixed costs are $674,000. Variable costs are $21.
41. Let x be the number of pairs of fenders manufactured. Then the cost functions in dollars are as follows:
    Steel: $C(x) = 260,000 + 5.26x$
    Aluminum: $C(x) = 385,000 + 12.67x$
    RMP: $C(x) = 95,000 + 13.19x$
    NPN: $C(x) = 95,000 + 9.53x$
43. $R = 2274x$
45. 2500
47. $R = 0.0704x$
49. $446, $127
51. $C = 209.03x + 447,917$
    $R = 266.67x$
    $P = 57.64x - 447,917$
53. $p = -5.6 + 155$
55. (a) $p \approx 300,000$ (b) 0.1 yen per ton
57. 11.76
59. Since no sales result in no revenue, we must have $R(0) = 0$.
61. The area is the base times the height. But the height is the price and the base is the number of items sold. Therefore, the area of the rectangle is the price times the number sold, which is the revenue.
63. $c = -6x + 12$
65. Let x be the number of cows, then the cost $C(x)$ in dollars is $C(x) = 13,386 + 393x$. The revenue function in dollars is $R(x) = 470x$. The profit function in dollars is $P(x) = 77x - 13,386$. The profit for an average of 97 cows is $P(97) = -5917$, that is a loss of $5917. For many such farms the property and buildings have already been paid off. Thus, the fixed costs for these farms are lower than stated in the table.
67. Let the cost revenue, and profits be given in thousands of dollars. We have
    $C(x) = 3.84x + 300,000$
    $R(x) = 4.8x$
    $P(x) = 0.96x - 300,000$

Number	200,000	300,000	400,000
Cost	1068	1452	1836
Revenue	960	1440	1920
Profit	−108	−12	84

EXERCISE SET 1.2

1. 2
3. 20
5. $(1.5, 4.5)$
7. $(1, 15)$
9. 2000
11. (a) Rental: $C_R = 320d$
    (b) Buy: $C_B = 28,000 + 40d$
    (c, d) 100 days. Renting.
13. 10 years
15. (a) Acme: $C_A = 75 + 0.40x$
    Bell: $C_B = 105 + 0.25x$
    (b) 200 miles. Acme
17. In-house
19. 20 bunches
21. $446. $446.
23. approximately 3925
25. approximately 174
27. steel
29. steel
31. (a) Boston (b) Houston
33. approximately 557,692
35. approximately 22,634
37. (a) The total cost for a manual machine is $C_m = 1000 + 16t$, where t is the processing time.

For an automatic machine the total cost is $C_a = 8000 + \frac{2n}{100}$.

(b) $x \approx 4430$

EXERCISE SET 1.3

1. $(x, y) = (2, 5)$
3. $(x, y) = (1, -2)$
5. $(x, y) = (-1, -1)$
7. $(x, y) = (-2, 1)$
9. $(x, y) = 1, 2)$
11. $(x, y, z) = (1, 2, 0)$
13. $(x, y, z) = (5, 0, 5)$
15. $(x, y, z) = (1, 2, 3)$
17. $(x, y, z) = (1.5, 1.5, -0.5)$
19. $(x, y, z) = (1, 1, 1)$
21. $(x, y, z, u) = (2, 1, 1, 2)$
23. $(x, y, z, u) = (1, -1, 2, 3)$
25. 5 quarters and 20 dimes
27. 10 nickles, 20 dimes, 6 quarters
29. 8 of type A, 6 of type B
31. $700 in the first bank, $300 in second
33. 160 style A, 300 of style B
35. Three of type A, five of type B, four of type C
37. 100 shares of MathOne, 20 shares of NewModule, 50 shares of JavaTime
39. 11 units of wood, 20 units of fabric, 1 unit of stuffing
41. Five batches of muffins, two batches of scones, three batches of croissants
43. Five units of iron, two units of calcium, eight units of folic acid
45. Three oranges, two cups of strawberries, one cup of blackberries
47. Since $x = -11$, there is no solution for the dietitian.
49. (a) Yes (b) additional 1000 of x-species

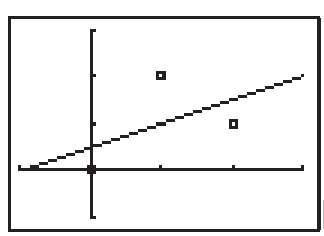
EXERCISE SET 1.4

1. Yes
3. Yes
5. No, the second row should be at the bottom.
7. $x = 2$ and $y = 3$
9. $(x, y) = (4 - 2t, t)$, where t is any number.
11. $(x, y, z, u) = (4 - 2s - 3t, 5 - 2s - 3t, s, t)$, where s and t are any numbers.
13. $(x, y, z, u) = (6 - 2s - 4t, 1 - 2s - 3t, s, t)$, where s and t are any numbers.
15. $(x, y, z, u, v, ww) = (r, 1 - 2s - 2t, s, 2t, t, 3)$, where r, s, and t are any numbers.
17. no solution
19. $(2t + 4, t)$
21. $(3t - 7, t)$
23. no solution
25. $(5 - \frac{7}{3}t, 10 - \frac{8}{3}t)$
27. $(3s + 2t - 6, s, t)$
29. $(5s - 3t - 7, s, t)$
31. $(1.5, -t - 0.5, t)$
33. no solution
35. $(1, t, t)$
37. $(x, y) = (1, 3)$
39. no solution
41. no solution
43. ???
45. no solution
47. no solution
49. nickels, dimes, quarters = $(3t - 8, 44 - 4t, t)$, where t is any natural number such that $3 \leq t \leq 11$
51. bonds, low-risk stocks, medium-risk stocks = $(7,900,000 - 2t, t)$ where $0 \leq t \leq 450,000$
53. 100 of style A, 150 of style B, 50 of style C
55. The production manager cannot meet the demands of his boss.
57. There are exactly two solutions: 9 of type A, 4 of type B, and none of type C, and 3 of type A, 5 of type B, and 4 of type C.
59. $([2c - 5a]t - 2d + 5b, [2a - c]t + d - 2b, t)$
61. $(x_1, x_2, x_3, x_4) = (700 - t, t - 100, 900 - t, t)$, where t is any number so that $100 \leq t \leq 700$.
63. The equilibrium is $(2, 3, 5, 6)$.
65. The first way saves one multiplication.

EXERCISE SET 1.5

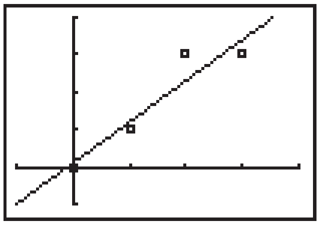
We used the TI-83 to obtain all of the following answers. See the Technology Corner for details.

1. (a) Linear regression gives $y = 0.50x + 0.50$, $r = 0.5000$.

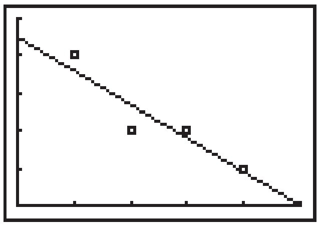


$[-1, 3] \times [-1, 3]$

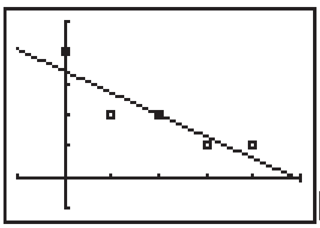
3. Linear regression gives $y = 1.1x + 0.1$, $r = 0.9467$.


$[-1, 4] \times [-1, 4]$

5. Linear regression gives $y = -0.9x + 4.5$, $r = -0.9234$.


$[0, 5] \times [0, 5]$

7. Linear regression gives $y = -0.7x + 3.4$, $r = -0.9037$.


$[-1, 5] \times [-1, 5]$

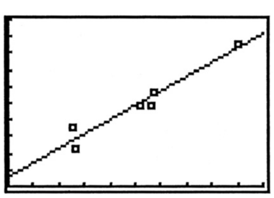
9. Linear regression gives $y = D(x) = -0.05266x + 4.8956$ and a price of 3.05 euros.

11. Linear regression gives $y = S(x) = 0.211x + 0.7503$ and 10658 mugs.

13. Linear regression gives $y = V(x) = -0.3346x + 4.01846$, value is \$4018.

15. (a) Linear regression gives $y = 2.0768x + 1.4086$, $r = 0.9961$.
(b) Using the value operation gives $y \approx 209$, when $x = 100$. Thus, the total cost is \$209,000.
(c) Inputting $y_2 = 125$ and using the intersect operation gives $x \approx 59.511$, which gives 59,511 dozen.

17. (a) Linear regression gives $y = 0.8605x + 0.5638$, $r = 0.9703$.


$[0, 10] \times [0, 10]$

(b) Using the value operation yields $y \approx 6.6$ when $x = 7$, giving GNP growth of 6.6%.
(c) Inputting $y_2 = 7$, graphing, and using the intersect property yields $x \approx 7.5$, giving a productivity growth of 7.5%.

19. Linear regression gives $y = -87.16x + 1343$, $r = -0.8216$.

21. (a) Linear regression gives $y = 3.4505x + 14.4455$, $r = 0.9706$.

(b) For each additional aphid per plant the percentage of times the virus is transmitted to the fruit increases by about 3.5% for the brown aphid and about 2% for the melon aphid.

(c) The brown aphid is more destructive since the brown aphid transmits the virus more often than the melon aphid.

EXERCISE REVIEW SET

1. $C = 2000 + 6x$
2. $R = 10x$
3. 500
4. 150
5. 1000
6. $p = -0.000005x + 1.5$
7. 12,000
8. $P = 21,000t - 70,000$
9. $P = 2000t - 20,000$
10. (a) $C_R = 150x$ (b) $C_O = 1400 - 10x$ (c) He should buy. (d) $x = 10$
11. $(x, y) = (1, 2)$
12. no solution
13. $(x, y) = (1 - 3t, t)$
14. $(x, y) = (3, 4)$
15. $(x, y, z) = (1.6 - 0.2t, 0.4t + 0.8, t)$
16. $(x, y, z) = (10 - t, t, 0)$
17. $(x, y, z) = (1, 3, 1)$
18. $(x, y, z) = (2t - 2, 7 - 3t, t)$
19. 3 par, 4 par, 5 par $= (t + 1, 8 - 2t, t)$
20. Seven standard sweepers, five deluxe sweepers
21. 20 ham specials, 30 beef specials
23. 90 liters of first solution, 60 of second, 150 of third
24. Plants 1, 2, and 3 should produce 100, 200, and 50, respectively.
25. $y = (5x + 1)/7$

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