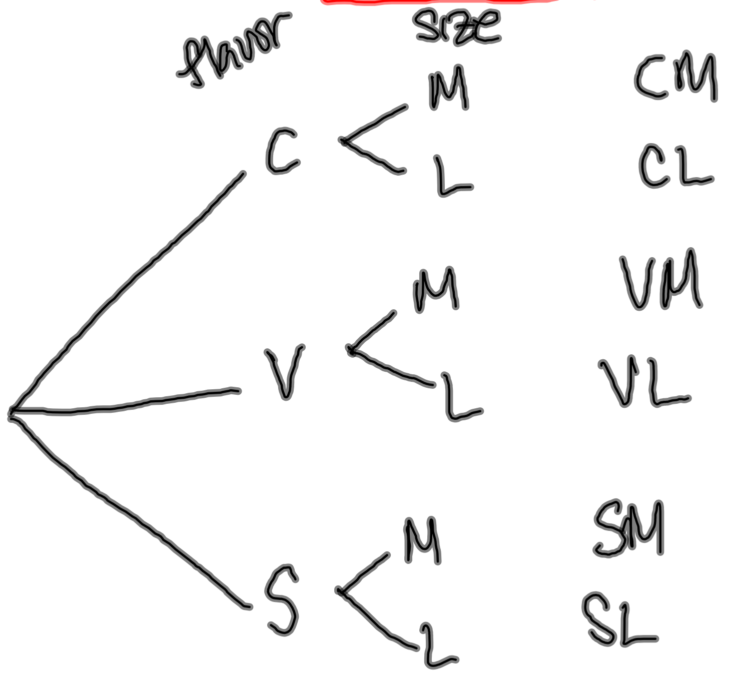
6.3 - The Multiplication Principle

Suppose a task T_1 can be done N_1 ways and a task T_2 can be done N_2 ways and so on until task T_n can be done N_n ways. Then the number of ways of performing the tasks
$T_1, T_2, \ldots T_n$ is given by the product

$$N_1 * N_2 \ldots * N_n$$

So if we have three kinds of yogurt with two sizes and five toppings the number of ways a yogurt can be chosen is

$$\underset{\text{flavor}}{\underline{3}} * \underset{\text{size}}{\underline{2}} * \underset{\text{top}}{\underline{5}} = 30$$



$S = \{CM, CL, VM, VL, SM, SL\}$, $n(S) = 6$

Example

Older Texas plates have 3 letters followed by two digit ~~numbers~~ and a letter. How many different plates are possible?

Answer

Draw a blank for each choice, $\underset{L}{\underline{26}} \cdot \underset{L}{\underline{26}} \ \underset{L}{\underline{26}} \cdot \underset{\#}{\underline{10}} \cdot \underset{\#}{\underline{10}} \cdot \underset{L}{\underline{26}}$

How many ways if no repeats are allowed? 175,697,600

$$\underset{L}{\underline{26}} \cdot \underset{L}{\underline{25}} \cdot \underset{L}{\underline{24}} \cdot \underset{\#}{\underline{10}} \cdot \underset{\#}{\underline{9}} \cdot \underset{L}{\underline{23}} = 32,292,000$$

6.4 - Permutations and Combinations

You want to arrange 10 students in a row of ~~of~~ 10 chairs.

$\underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 10!$

$= 3{,}628{,}800$

FACTORIALS: $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ and $0! = 1$

Example

We have 10 students again, but only 4 chairs. How many ways can we seat the 10 students if we want to seat 4 of them in the chairs?

$\underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} = 5040$

PERMUTATIONS. If we have a set of n elements and we want to take r of them in an arrangement, we say the number of permutations of n things taken r at a time is $P(n, r)$.

In our second example we had 10 students taken 4 at a time, so

$$10\cdot 9\cdot 8\cdot 7\cdot \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{10!}{6!} = \frac{10!}{(10-4)!}$$

10 nPr 4

In general, $\boxed{P(n,r) = \frac{n!}{(n-r)!}}$ in your calculator it is nPr

Example

How many ways can gold, silver and bronze medals be awarded in a race of 12 people?

$$\underset{G}{\underline{12}} \cdot \underset{S}{\underline{11}} \cdot \underset{B}{\underline{10}} = 1320$$

$P(12,3) = 12\text{nPr}3$

What if the n objects contain some that are identical?

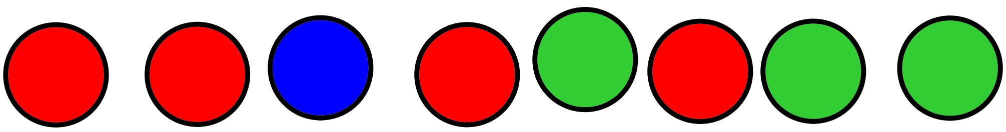
Count only the DISTINGUISHABLE permutations (the ones that look different).

If there are n_1 items of type 1 and n_2 items of type 2 and ... n_r items of type r, then the number of distinguishable permutations of the $n = n_1 + n_2 + \ldots + n_r$ items is

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Example

We have 4 red marbles, 3 green marbles and one blue marble. How many distinguishable permutations of the 8 marbles are there?



$$\frac{8!}{\underbrace{4!}_{R}\,\underbrace{3!}_{G}\,\underbrace{1!}_{B}} = 280$$

"words" are arrangements of letters.

How many ways can we choose a *group* of 4 students the 10 students?

How many ways we can arrange them? $10 \cdot 9 \cdot 8 \cdot 7$

How many ways can the group be re-arranged? $4 \cdot 3 \cdot 2 \cdot 1$

So the number of ways a group of 4 can be chosen is

$$\frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210 = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{4!\,6!}$$

$$= C(10,4) = 10 \text{ nCr } 4$$

This is the number of *combinations* of 10 items taken 4 at a time.

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Example
Suppose a group of 7 workers decides to send a delegation of 2 to their supervisor.

How many delegations are possible?

$$\frac{7 \cdot 6}{2} = 21 = C(7,2) = 7nCr2$$

If there are two women and 5 men in the group of workers, how many delegations will include at least one woman?

1W and 1M or 2W and 0M

$$2 \cdot 5 + 1 \cdot 1$$

$$\underset{1W}{\underline{C(2,1)}} \cdot \underset{1M}{\underline{C(5,1)}} + \underset{2W}{\underline{C(2,2)}} \cdot \underset{0M}{\underline{C(5,0)}} = 2 \cdot 5 + 1 \cdot 1 = 11$$

$B_1\ B_2\ B_3\ B_4\ B_5$

Example

A bag contains 5 blue, 1 green and 3 orange jelly beans. You choose 3 at random. How many $\boxed{\text{samples}}$ are possible in which

a) the jelly beans are all blue?

$\underset{3B}{C(5,3)} \cdot \underset{0B^c}{C(4,0)} = \quad 5nCr3 = 10$

b) the jelly beans are all green? $1nCr3$

0

c) the jelly beans are all orange? $1 = \underset{3O}{C(3,3)} \cdot \underset{0O^c}{C(6,0)}$

d) there are 2 blue and 1 green?

$\underset{2B}{C(5,2)} \cdot \underset{1G}{C(1,1)} \cdot \underset{0O}{C(3,0)} = 10 \cdot 1 \cdot 1 = 10$

e) there are 2 blue and 1 orange?

$\underset{2B}{C(5,2)} \cdot \underset{1O}{C(3,1)} \cdot \underset{0G}{C(1,0)} = 10 \cdot 3 \cdot 1 = 30$

f) How many ways to choose 3 jelly beans?

$C(9,3) = 9nCr3 = 84 \longrightarrow$ $0B$] 4, $1B$, $2B$, $3B$] 80

g) How many ways to choose ~~no blue~~?

$\underset{0B}{C(5,0)} \cdot \underset{3B^c}{C(4,3)} = 4$

h) How many ways to choose at least one blue?

1B and $2B^c$ + 2B and $1B^c$ + 3B and $0B^c$

$\underset{1B}{C(5,1)} \cdot \underset{2B^c}{C(4,2)} + \underset{2B}{C(5,2)} \cdot \underset{1B^c}{C(4,1)} + \underset{3B}{C(5,3)} \cdot \underset{1B^c}{C(4,0)}$

$5 \cdot 6 + 10 \cdot 4 + 10 \cdot 1 = 80$

arrangement of digts $123 \neq 312$

A social security number has 9 digits. How many are possible with no restrictions?

$\underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = 10^9$

How many are possible if the first three digits are not zero?

$\underline{9} \cdot \underline{9} \cdot \underline{9} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} \cdot \underline{10} = 729{,}000{,}000$

How many ways can 6 of 54 numbers be chosen if order doesn't matter?

$$C(54,6) = 25{,}827{,}165$$

How many ways to choose no winning numbers?

$$\underbrace{C(6,0)}_{\text{0 win}} \cdot \underbrace{C(48,6)}_{\text{6 Losing}} = 12{,}271{,}512$$

How many ways to choose at least 3 winning numbers?

3W and 3L or 4W and 2L or 5W and 1L or 6W and 0L

$$\underbrace{C(6,3)}_{3W} \cdot \underbrace{C(48,3)}_{3L} + \underbrace{C(6,4)}_{4W} \cdot \underbrace{C(48,2)}_{2L} + \underbrace{C(6,5)}_{5W} \cdot \underbrace{C(48,1)}_{1L} + \underbrace{C(6,6)}_{6W} \cdot \underbrace{C(48,0)}_{0L}$$

$$20 \cdot 17296 + 15 \cdot 1128 + 6 \cdot 48 + 1 \cdot 1$$

$$345{,}920 + 16920 + 288 + 1$$

$$= 363{,}129$$

10 Good

A bag contains 12 oranges and 2 are rotten. Take a sample of 3.

How many ways to choose the sample of 3?

$C(12,3) = 220$

How many samples have 0 rotten oranges?

$$\underbrace{C(2,0)}_{0R} \cdot \underbrace{C(10,3)}_{3G} = 1 \cdot 120 = \underline{120}$$

How many samples have 1 rotten orange?

$$\underbrace{C(2,1)}_{1R} \cdot \underbrace{C(10,2)}_{2G} = 2 \cdot 45 = \underline{90}$$

How many samples have 2 rotten oranges?

$$\underbrace{C(2,2)}_{2R} \cdot \underbrace{C(10,1)}_{1G} = 1 \cdot 10 = \underline{10}$$

How many samples have 3 rotten oranges? 0

COMBINATION

How many ways can a $\underbrace{\text{hand}}$ of 6 clubs be chosen from a standard deck?

$$\underset{\text{6 Clubs}}{C(13,6)} \cdot \underset{0\ C^c}{C(39,0)} = 1716$$

From 10 names on a ballot, how many ways can 4 be elected to a committee if each has a different responsibility?

$$\underset{P}{10} \cdot \underset{VP}{9} \cdot \underset{T}{8} \cdot \underset{H}{7} = 5040$$

$$\left(P(10,4)\right)$$

A sales person has 6 prospects. How many ways can she arrange her schedule to see all 6?

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6! = 720$$

$$P(6,6)$$

$13! = 6,227,020,800$

You have a group of 13 different books. Three are math books, four are chemistry and six are history books. ~~How many different arrangements~~ arrangements are possible if books of the same type are kept together?

$\boxed{M_1 \quad M_2 \quad M_3}$ $\boxed{C_1 \; C_4 \; C_2 \; C_3}$ $\boxed{H_1 \; H_2 \; H_3 \; H_4 \; H_5 \; H_6}$

$$\underbrace{3 \cdot 2 \cdot 1}_{\text{arr. groups}} \cdot \underbrace{3 \cdot 2 \cdot 1}_{\text{MATH}} \cdot \underbrace{4 \cdot 3 \cdot 2 \cdot 1}_{\text{CHEM}} \cdot \underbrace{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}_{\text{HIST.}}$$

$= 622,080$