SYSTEMS OF LINEAR EQUATIONS AND MATRICES

2

T HE LINEAR EQUATIONS in two variables studied in Chapter 1 are readily extended to the case involving more than two variables. For example, a linear equation in three variables represents a plane in three-dimensional space. In this chapter, we see how some real-world problems can be formulated in terms of systems of linear equations, and we also develop two methods for solving these equations.

In addition, we see how *matrices* (rectangular arrays of numbers) can be used to write systems of linear equations in compact form. We then go on to consider some real-life applications of matrices. Finally, we show how matrices can be used to describe the Leontief input–output model, an important tool used by economists. For his work in formulating this model, Wassily Leontief was awarded the Nobel Prize in 1973.

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Checkers Rent-A-Car is planning to expand its fleet of cars next quarter. How should they use their budget of $12 million to meet the expected additional demand for compact and full-size cars? In Example 5, page 133, we will see how we can find the solution to this problem by solving a system of equations.

2.1 Systems of Linear Equations: An Introduction

Systems of Equations

Recall that in Section 1.4 we had to solve two simultaneous linear equations in order to find the *break-even point* and the *equilibrium point*. These are two examples of real-world problems that call for the solution of a **system of linear equations** in two or more variables. In this chapter we take up a more systematic study of such systems.

We begin by considering a system of two linear equations in two variables. Recall that such a system may be written in the general form

$$ax + by = h$$
$$cx + dy = k \tag{1}$$

where a, b, c, d, h, and k are real constants and neither a and b nor c and d are both zero.

Now let's study the nature of the **solution of a system of linear equations** in more detail. Recall that the graph of each equation in System (1) is a straight line in the plane, so that geometrically the solution to the system is the point(s) of intersection of the two straight lines L_1 and L_2, represented by the first and second equations of the system.

Given two lines L_1 and L_2, *one and only one* of the following may occur:

a. L_1 and L_2 intersect at exactly one point.
b. L_1 and L_2 are parallel and coincident.
**c.** L_1 and L_2 are parallel and distinct.

(See Figure 1.) In the first case, the system has a unique solution corresponding to the single point of intersection of the two lines. In the second case, the system has infinitely many solutions corresponding to the points lying on the same line. Finally, in the third case, the system has no solution because the two lines do not intersect.
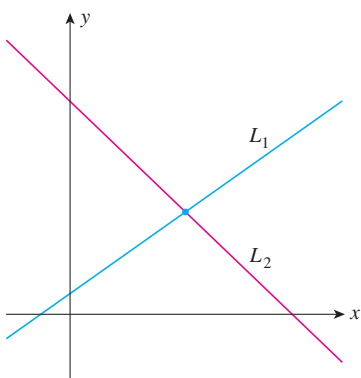
FIGURE **1**
(a) Unique solution
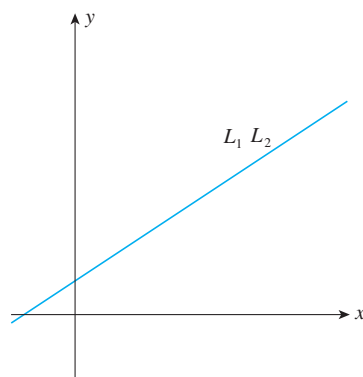
(b) Infinitely many solutions
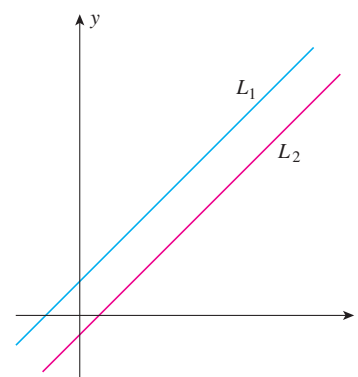
(c) No solution

Explore & Discuss

Generalize the discussion on this page to the case where there are three straight lines in the plane defined by three linear equations. What if there are n lines defined by n equations?

Let's illustrate each of these possibilities by considering some specific examples.

1. **A system of equations with exactly one solution** Consider the system

$$2x - y = 1$$
$$3x + 2y = 12$$

Solving the first equation for y in terms of x, we obtain the equation

$$y = 2x - 1$$

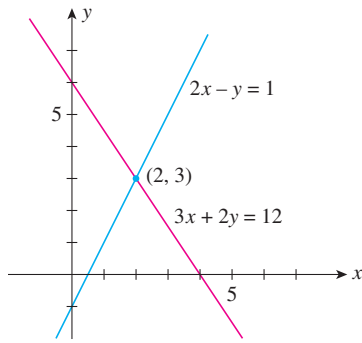Substituting this expression for y into the second equation yields

$$3x + 2(2x - 1) = 12$$
$$3x + 4x - 2 = 12$$
$$7x = 14$$
$$x = 2$$

Finally, substituting this value of x into the expression for y obtained earlier gives

$$y = 2(2) - 1 = 3$$

Therefore, the unique solution of the system is given by $x = 2$ and $y = 3$. Geometrically, the two lines represented by the two linear equations that make up the system intersect at the point $(2, 3)$ (Figure 2).

FIGURE 2
A system of equations with one solution

Note We can check our result by substituting the values $x = 2$ and $y = 3$ into the equations. Thus,

$$2(2) - (3) = 1 \quad \checkmark$$
$$3(2) + 2(3) = 12 \quad \checkmark$$

From the geometric point of view, we have just verified that the point $(2, 3)$ lies on both lines. ∎

2. **A system of equations with infinitely many solutions** Consider the system

$$2x - y = 1$$
$$6x - 3y = 3$$

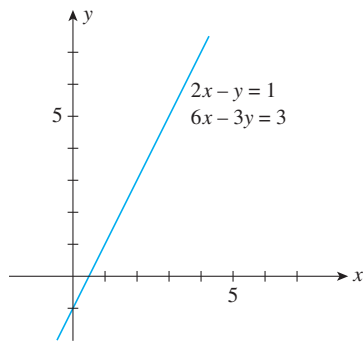Solving the first equation for y in terms of x, we obtain the equation

$$y = 2x - 1$$

Substituting this expression for y into the second equation gives

$$6x - 3(2x - 1) = 3$$
$$6x - 6x + 3 = 3$$
$$0 = 0$$

which is a true statement. This result follows from the fact that the second equation is equivalent to the first. (To see this, just multiply both sides of the first equation by 3.) Our computations have revealed that the system of two equations is equivalent to the single equation $2x - y = 1$. Thus, any ordered pair of numbers (x, y) satisfying the equation $2x - y = 1$ (or $y = 2x - 1$) constitutes a solution to the system.

In particular, by assigning the value t to x, where t is any real number, we find that $y = 2t - 1$ and so the ordered pair $(t, 2t - 1)$ is a solution of the system. The variable t is called a **parameter.** For example, setting $t = 0$ gives the point $(0, -1)$ as a solution of the system, and setting $t = 1$ gives the point $(1, 1)$ as another solution. Since t represents any real number, there are infinitely many solutions of the

FIGURE 3
A system of equations with infinitely many solutions; each point on the line is a solution.
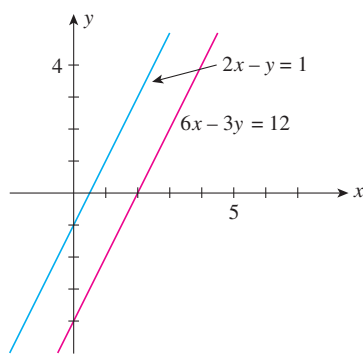
FIGURE 4
A system of equations with no solution

system. Geometrically, the two equations in the system represent the same line, and all solutions of the system are points lying on the line (Figure 3). Such a system is said to be **dependent.**

3. **A system of equations that has no solution** Consider the system

$$2x - y = 1$$
$$6x - 3y = 12$$

The first equation is equivalent to $y = 2x - 1$. Substituting this expression for y into the second equation gives

$$6x - 3(2x - 1) = 12$$
$$6x - 6x + 3 = 12$$
$$0 = 9$$

which is clearly impossible. Thus, there is no solution to the system of equations. To interpret this situation geometrically, cast both equations in the slope-intercept form, obtaining

$$y = 2x - 1$$
$$y = 2x - 4$$

We see at once that the lines represented by these equations are parallel (each has slope 2) and distinct since the first has y-intercept -1 and the second has y-intercept -4 (Figure 4). Systems with no solutions, such as this one, are said to be **inconsistent.**

Explore & Discuss

1. Consider a system composed of two linear equations in two variables. Can the system have exactly two solutions? Exactly three solutions? Exactly a finite number of solutions?

2. Suppose at least one of the equations in a system composed of two equations in two variables is nonlinear. Can the system have no solution? Exactly one solution? Exactly two solutions? Exactly a finite number of solutions? Infinitely many solutions? Illustrate each answer with a sketch.

Note   We have used the method of substitution in solving each of these systems. If you are familiar with the method of elimination, you might want to re-solve each of these systems using this method. We will study the method of elimination in detail in Section 2.2.   ■

In Section 1.4, we presented some real-world applications of systems involving two linear equations in two variables. Here is an example involving a system of three linear equations in three variables.

APPLIED EXAMPLE 1 Manufacturing: Production Scheduling Ace Novelty wishes to produce three types of souvenirs: types A, B, and C. To manufacture a type-A souvenir requires 2 minutes on machine I, 1 minute on machine II, and 2 minutes on machine III. A type-B souvenir requires 1 minute on machine I, 3 minutes on machine II, and 1 minute on machine III. A type-C souvenir requires 1 minute on machine I and 2 minutes each on machines II and III. There are 3 hours available on machine I, 5 hours available on machine II, and 4 hours available on machine III for processing the order. How many sou-

venirs of each type should Ace Novelty make in order to use all of the available time? Formulate but do not solve the problem. (We will solve this problem in Example 7, Section 2.2.)

Solution   The given information may be tabulated as follows:

	Type A	Type B	Type C	Time Available (min)
Machine I	2	1	1	180
Machine II	1	3	2	300
Machine III	2	1	2	240

We have to determine the number of each of *three* types of souvenirs to be made. So, let x, y, and z denote the respective numbers of type-A, type-B, and type-C souvenirs to be made. The total amount of time that machine I is used is given by $2x + y + z$ minutes and must equal 180 minutes. This leads to the equation

$$2x + y + z = 180 \qquad \text{\color{magenta}Time spent on machine I}$$

Similar considerations on the use of machines II and III lead to the following equations:

$$x + 3y + 2z = 300 \qquad \text{\color{magenta}Time spent on machine II}$$
$$2x + y + 2z = 240 \qquad \text{\color{magenta}Time spent on machine III}$$

Since the variables x, y, and z must satisfy simultaneously the three conditions represented by the three equations, the solution to the problem is found by solving the following system of linear equations:

$$
\begin{aligned}
2x + y + z &= 180 \\
x + 3y + 2z &= 300 \\
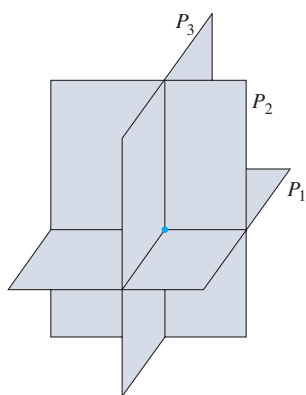2x + y + 2z &= 240
\end{aligned}
$$
◼

Solutions of Systems of Equations

We will complete the solution of the problem posed in Example 1 later on (page 84). For the moment, let's look at the geometric interpretation of a system of linear equations, such as the system in Example 1, in order to gain some insight into the nature of the solution.

A linear system composed of three linear equations in three variables x, y, and z has the general form

$$
\begin{aligned}
a_1x + b_1y + c_1z &= d_1 \\
a_2x + b_2y + c_2z &= d_2 \\
a_3x + b_3y + c_3z &= d_3
\end{aligned}
\qquad (2)
$$

Just as a linear equation in two variables represents a straight line in the plane, it can be shown that a linear equation $ax + by + cz = d$ (a, b, and c not all equal to zero) in three variables represents a plane in three-dimensional space. Thus, each equation in System (2) represents a *plane* in three-dimensional space, and the *solution(s) of the system* is precisely the point(s) of intersection of the three planes defined by the three linear equations that make up the system. As before, the system has one and only one solution, infinitely many solutions, or no solution, depending on whether and how the planes intersect one another. Figure 5 illustrates each of these possibilities.

In Figure 5a, the three planes intersect at a point corresponding to the situation in which System (2) has a unique solution. Figure 5b depicts a situation in which there are infinitely many solutions to the system. Here, the three planes intersect along a line, and the solutions are represented by the infinitely many points lying on this line. In Figure 5c, the three planes are parallel and distinct, so there is no point in common to all three planes; System (2) has no solution in this case.

FIGURE 5
(a) A unique solution

(b) Infinitely many solutions

(c) No solution

Note   The situations depicted in Figure 5 are by no means exhaustive. You may consider various other orientations of the three planes that would illustrate the three possible outcomes in solving a system of linear equations involving three variables.

> ### Linear Equations in n Variables
>
> A linear equation in n variables, $x_1, x_2, \ldots, x_n$ is an equation of the form
>
> $$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = c$$
>
> where $a_1, a_2, \ldots, a_n$ (not all zero) and c are constants.

For example, the equation

$$3x_1 + 2x_2 - 4x_3 + 6x_4 = 8$$

is a linear equation in the four variables, $x_1, x_2, x_3,$ and x_4.

When the number of variables involved in a linear equation exceeds three, we no longer have the geometric interpretation we had for the lower-dimensional spaces. Nevertheless, the algebraic concepts of the lower-dimensional spaces generalize to higher dimensions. For this reason, a linear equation in n variables, $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = c$, where $a_1, a_2, \ldots, a_n$ are not all zero, is referred to as an *n-dimensional hyperplane*. We may interpret the solution(s) to a system comprising a finite number of such linear equations to be the *point(s) of intersection* of the hyperplanes defined by the equations that make up the system. As in the case of systems involving two or three variables, it can be shown that only three possibilities exist regarding the nature of the solution of such a system: (1) a unique solution, (2) infinitely many solutions, or (3) no solution.

Explore & Discuss

Refer to the Note on page 70.

Using the orientation of three planes, illustrate the outcomes in solving a system of three linear equations in three variables that result in no solution or infinitely many solutions.

2.1 Self-Check Exercises

1. Determine whether the system of linear equations

$$2x - 3y = 12$$
$$x + 2y = 6$$

has (a) a unique solution, (b) infinitely many solutions, or (c) no solution. Find all solutions whenever they exist. Make a sketch of the set of lines described by the system.

2. A farmer has 200 acres of land suitable for cultivating crops A, B, and C. The cost per acre of cultivating crops A, B, and C is $40, $60, and $80, respectively. The farmer has

$12,600 available for cultivation. Each acre of crop A requires 20 labor-hours, each acre of crop B requires 25 labor-hours, and each acre of crop C requires 40 labor-hours. The farmer has a maximum of 5950 labor-hours available. If she wishes to use all of her cultivatable land, the entire budget, and all the labor available, how many acres of each crop should she plant? Formulate but do not solve the problem.

Solutions to Self-Check Exercises 2.1 can be found on page 75.

2.1 Concept Questions

1. Suppose you are given a system of two linear equations in two variables.
   a. What can you say about the solution(s) of the system of equations?
   b. Give a geometric interpretation of your answers to the question in part (a).

2. Suppose you are given a system of two linear equations in two variables.
   a. Explain what it means for the system to be dependent.
   b. Explain what it means for the system to be inconsistent.

2.1 Exercises

In Exercises 1–12, determine whether each system of linear equations has (a) one and only one solution, (b) infinitely many solutions, or (c) no solution. Find all solutions whenever they exist.

1. $x - 3y = -1$
    $4x + 3y = 11$

2. $2x - 4y = 5$
    $3x + 2y = 6$

3. $x + 4y = 7$
    $\frac{1}{2}x + 2y = 5$

4. $3x - 4y = 7$
    $9x - 12y = 14$

5. $x + 2y = 7$
    $2x - y = 4$

6. $\frac{3}{2}x - 2y = 4$
    $x + \frac{1}{3}y = 2$

7. $2x - 5y = 10$
    $6x - 15y = 30$

8. $5x - 6y = 8$
    $10x - 12y = 16$

9. $4x - 5y = 14$
    $2x + 3y = -4$

10. $\frac{5}{4}x - \frac{2}{3}y = 3$
     $\frac{1}{4}x + \frac{5}{3}y = 6$

11. $2x - 3y = 6$
     $6x - 9y = 12$

12. $\frac{2}{3}x + y = 5$
     $\frac{1}{2}x + \frac{3}{4}y = \frac{15}{4}$

13. Determine the value of k for which the system of linear equations

$$2x - y = 3$$
$$4x + ky = 4$$

has no solution.

14. Determine the value of k for which the system of linear equations

$$3x + 4y = 12$$
$$x + ky = 4$$

has infinitely many solutions. Then find all solutions corresponding to this value of k.

In Exercises 15–27, formulate but do not solve the problem. You will be asked to solve these problems in the next section.

15. AGRICULTURE The Johnson Farm has 500 acres of land allotted for cultivating corn and wheat. The cost of cultivating corn and wheat (including seeds and labor) is $42 and $30 per acre, respectively. Jacob Johnson has $18,600 available for cultivating these crops. If he wishes to use all the allotted land and his entire budget for cultivating these two crops, how many acres of each crop should he plant?

16. **INVESTMENTS** Michael Perez has a total of $2000 on deposit with two savings institutions. One pays interest at the rate of 6%/year, whereas the other pays interest at the rate of 8%/year. If Michael earned a total of $144 in interest during a single year, how much does he have on deposit in each institution?

17. **MIXTURES** The Coffee Shoppe sells a coffee blend made from two coffees, one costing $5/lb and the other costing $6/lb. If the blended coffee sells for $5.60/lb, find how much of each coffee is used to obtain the desired blend. Assume that the weight of the blended coffee is 100 lb.

18. **INVESTMENTS** Kelly Fisher has a total of $30,000 invested in two municipal bonds that have yields of 8% and 10% interest per year, respectively. If the interest Kelly receives from the bonds in a year is $2640, how much does she have invested in each bond?

19. **RIDERSHIP** The total number of passengers riding a certain city bus during the morning shift is 1000. If the child's fare is $.50, the adult fare is $1.50, and the total revenue from the fares in the morning shift is $1300, how many children and how many adults rode the bus during the morning shift?

20. **REAL ESTATE** Cantwell Associates, a real estate developer, is planning to build a new apartment complex consisting of one-bedroom units and two- and three-bedroom townhouses. A total of 192 units is planned, and the number of family units (two- and three-bedroom townhouses) will equal the number of one-bedroom units. If the number of one-bedroom units will be 3 times the number of three-bedroom units, find how many units of each type will be in the complex.

21. **INVESTMENT PLANNING** The annual returns on Sid Carrington's three investments amounted to $21,600: 6% on a savings account, 8% on mutual funds, and 12% on bonds. The amount of Sid's investment in bonds was twice the amount of his investment in the savings account, and the interest earned from his investment in bonds was equal to the dividends he received from his investment in mutual funds. Find how much money he placed in each type of investment.

22. **INVESTMENT CLUB** A private investment club has $200,000 earmarked for investment in stocks. To arrive at an acceptable overall level of risk, the stocks that management is considering have been classified into three categories: high risk, medium risk, and low risk. Management estimates that high-risk stocks will have a rate of return of 15%/year; medium-risk stocks, 10%/year; and low-risk stocks, 6%/year. The members have decided that the investment in low-risk stocks should be equal to the sum of the investments in the stocks of the other two categories. Determine how much the club should invest in each type of stock if the investment goal is to have a return of $20,000/year on the total investment. (Assume that all the money available for investment is invested.)

23. **MIXTURE PROBLEM—FERTILIZER** Lawnco produces three grades of commercial fertilizers. A 100-lb bag of grade-A fertilizer contains 18 lb of nitrogen, 4 lb of phosphate, and 5 lb of potassium. A 100-lb bag of grade-B fertilizer contains 20 lb of nitrogen and 4 lb each of phosphate and potassium. A 100-lb bag of grade-C fertilizer contains 24 lb of nitrogen, 3 lb of phosphate, and 6 lb of potassium. How many 100-lb bags of each of the three grades of fertilizers should Lawnco produce if 26,400 lb of nitrogen, 4900 lb of phosphate, and 6200 lb of potassium are available and all the nutrients are used?

24. **BOX-OFFICE RECEIPTS** A theater has a seating capacity of 900 and charges $4 for children, $6 for students, and $8 for adults. At a certain screening with full attendance, there were half as many adults as children and students combined. The receipts totaled $5600. How many children attended the show?

25. **MANAGEMENT DECISIONS** The management of Hartman Rent-A-Car has allocated $1.5 million to buy a fleet of new automobiles consisting of compact, intermediate-size, and full-size cars. Compacts cost $12,000 each, intermediate-size cars cost $18,000 each, and full-size cars cost $24,000 each. If Hartman purchases twice as many compacts as intermediate-size cars and the total number of cars to be purchased is 100, determine how many cars of each type will be purchased. (Assume that the entire budget will be used.)

26. **INVESTMENT CLUBS** The management of a private investment club has a fund of $200,000 earmarked for investment in stocks. To arrive at an acceptable overall level of risk, the stocks that management is considering have been classified into three categories: high risk, medium risk, and low risk. Management estimates that high-risk stocks will have a rate of return of 15%/year; medium-risk stocks, 10%/year; and low-risk stocks, 6%/year. The investment in low-risk stocks is to be twice the sum of the investments in stocks of the other two categories. If the investment goal is to have an average rate of return of 9%/year on the total investment, determine how much the club should invest in each type of stock. (Assume that all the money available for investment is invested.)

27. **DIET PLANNING** A dietitian wishes to plan a meal around three foods. The percentage of the daily requirements of proteins, carbohydrates, and iron contained in each ounce of the three foods is summarized in the following table:

	Food I	Food II	Food III
Proteins (%)	10	6	8
Carbohydrates (%)	10	12	6
Iron (%)	5	4	12

Determine how many ounces of each food the dietitian should include in the meal to meet exactly the daily requirement of proteins, carbohydrates, and iron (100% of each).

In Exercises 28–30, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

28. A system composed of two linear equations must have at least one solution if the straight lines represented by these equations are nonparallel.

29. Suppose the straight lines represented by a system of three linear equations in two variables are parallel to each other. Then the system has no solution or it has infinitely many solutions.

30. If at least two of the three lines represented by a system composed of three linear equations in two variables are parallel, then the system has no solution.

2.1 Solutions to Self-Check Exercises

1. Solving the first equation for y in terms of x, we obtain

$$y = \frac{2}{3}x - 4$$

Next, substituting this result into the second equation of the system, we find

$$x + 2\left(\frac{2}{3}x - 4\right) = 6$$

$$x + \frac{4}{3}x - 8 = 6$$

$$\frac{7}{3}x = 14$$

$$x = 6$$

Substituting this value of x into the expression for y obtained earlier, we have

$$y = \frac{2}{3}(6) - 4 = 0$$

Therefore, the system has the unique solution $x = 6$ and $y = 0$. Both lines are shown in the accompanying figure.
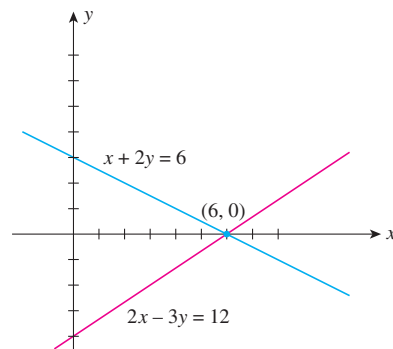
2. Let x, y, and z denote the number of acres of crop A, crop B, and crop C, respectively, to be cultivated. Then, the condition that all the cultivatable land be used translates into the equation

$$x + y + z = 200$$

Next, the total cost incurred in cultivating all three crops is $40x + 60y + 80z$ dollars, and since the entire budget is to be expended, we have

$$40x + 60y + 80z = 12{,}600$$

Finally, the amount of labor required to cultivate all three crops is $20x + 25y + 40z$ hr, and since all the available labor is to be used, we have

$$20x + 25y + 40z = 5950$$

Thus, the solution is found by solving the following system of linear equations:

$$
\begin{aligned}
x + y + z &= 200 \\
40x + 60y + 80z &= 12{,}600 \\
20x + 25y + 40z &= 5{,}950
\end{aligned}
$$

2.2 Systems of Linear Equations: Unique Solutions

The Gauss–Jordan Method

The method of substitution used in Section 2.1 is well suited to solving a system of linear equations when the number of linear equations and variables is small. But for large systems, the steps involved in the procedure become difficult to manage.

The **Gauss–Jordan elimination method** is a suitable technique for solving systems of linear equations of any size. One advantage of this technique is its adaptability to the computer. This method involves a sequence of operations on a system of linear equations to obtain at each stage an **equivalent system**—that is, a system having the same solution as the original system. The reduction is complete when the original system has been transformed so that it is in a certain standard form from which the solution can be easily read.

The operations of the Gauss–Jordan elimination method are

1. Interchange any two equations.
2. Replace an equation by a nonzero constant multiple of itself.
3. Replace an equation by the sum of that equation and a constant multiple of any other equation.

To illustrate the Gauss–Jordan elimination method for solving systems of linear equations, let's apply it to the solution of the following system:

$$2x + 4y = 8$$
$$3x - 2y = 4$$

We begin by working with the first, or x, column. First, we transform the system into an equivalent system in which the coefficient of x in the first equation is 1:

$$\begin{aligned} 2x + 4y &= 8 \\ 3x - 2y &= 4 \end{aligned} \tag{3a}$$

$$\begin{aligned} x + 2y &= 4 \\ 3x - 2y &= 4 \end{aligned} \qquad \text{Multiply the first equation in (3a) by } \tfrac{1}{2} \text{ (operation 2).} \tag{3b}$$

Next, we eliminate x from the second equation:

$$\begin{aligned} x + 2y &= 4 \\ -8y &= -8 \end{aligned}$$

Replace the second equation in (3b) by the sum of $-3 \times$ the first equation + the second equation (operation 3): $\tag{3c}$

$$\begin{aligned} -3x - 6y &= -12 \\ \underline{3x - 2y} &= \underline{4} \\ -8y &= -8 \end{aligned}$$

Then, we obtain the following equivalent system in which the coefficient of y in the second equation is 1:

$$\begin{aligned} x + 2y &= 4 \\ y &= 1 \end{aligned} \qquad \text{Multiply the second equation in (3c) by } -\tfrac{1}{8} \text{ (operation 2).} \tag{3d}$$

Next, we eliminate y in the first equation:

$$\begin{aligned} x &= 2 \\ y &= 1 \end{aligned}$$

Replace the first equation in (3d) by the sum of $-2 \times$ the second equation + the first equation (operation 3):

$$\begin{aligned} x + 2y &= 4 \\ \underline{-2y} &= \underline{-2} \\ x &= 2 \end{aligned}$$

This system is now in standard form, and we can read off the solution to System (3a) as $x = 2$ and $y = 1$. We can also express this solution as $(2, 1)$ and interpret it geometrically as the point of intersection of the two lines represented by the two linear equations that make up the given system of equations.

Let's consider another example, involving a system of three linear equations and three variables.

EXAMPLE 1 Solve the following system of equations:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27$$
$$-x + \ y + 2z = \ 2$$

Solution   First, we transform this system into an equivalent system in which the coefficient of x in the first equation is 1:

$$2x + 4y + 6z = 22$$
$$3x + 8y + 5z = 27 \tag{4a}$$
$$-x + \ y + 2z = \ 2$$

$$x + 2y + 3z = 11$$
$$3x + 8y + 5z = 27 \qquad \text{Multiply the first equation} \atop \text{in (4a) by } \frac{1}{2}. \tag{4b}$$
$$-x + \ y + 2z = \ 2$$

Next, we eliminate the variable x from all equations except the first:

$$x + 2y + 3z = \ 11 \qquad \text{Replace the second equation in (4b)}$$
$$2y - 4z = -6 \qquad \text{by the sum of } -3 \times \text{the first equa-}$$
$$-x + \ y + 2z = \ 2 \qquad \text{tion} + \text{the second equation:} \tag{4c}$$

$$\begin{array}{r} -3x - 6y - 9z = -33 \\ 3x + 8y + 5z = \ \ \ 27 \\ \hline 2y - 4z = \ -6 \end{array}$$

$$x + 2y + 3z = \ 11 \qquad \text{Replace the third equation in (4c)}$$
$$2y - 4z = -6 \qquad \text{by the sum of the first equation} +$$
$$3y + 5z = \ 13 \qquad \text{the third equation:} \tag{4d}$$

$$\begin{array}{r} x + 2y + 3z = 11 \\ -x + \ y + 2z = \ 2 \\ \hline 3y + 5z = 13 \end{array}$$

Then we transform System (4d) into yet another equivalent system, in which the coefficient of y in the second equation is 1:

$$x + 2y + 3z = \ 11$$
$$y - 2z = -3 \qquad \text{Multiply the second equation} \atop \text{in (4d) by } \frac{1}{2}. \tag{4e}$$
$$3y + 5z = \ 13$$

We now eliminate y from all equations except the second, using operation 3 of the elimination method:

$$x \qquad + 7z = \ 17 \qquad \text{Replace the first equation in (4e)}$$
$$y - 2z = -3 \qquad \text{by the sum of the first equation} +$$
$$3y + 5z = \ 13 \qquad (-2) \times \text{the second equation:} \tag{4f}$$

$$\begin{array}{r} x + 2y + 3z = 11 \\ -2y + 4z = \ 6 \\ \hline x \qquad + 7z = 17 \end{array}$$

$$x \qquad + 7z = \ 17 \qquad \text{Replace the third equation in (4f)}$$
$$y - 2z = -3 \qquad \text{by the sum of } (-3) \times \text{the second}$$
$$11z = \ 22 \qquad \text{equation} + \text{the third equation:} \tag{4g}$$

$$\begin{array}{r} -3y + \ 6z = \ 9 \\ 3y + \ 5z = 13 \\ \hline 11z = 22 \end{array}$$

Multiplying the third equation by $\frac{1}{11}$ in (4g) leads to the system

$$
\begin{aligned}
x \quad\quad + 7z &= 17 \\
y - 2z &= -3 \\
z &= 2
\end{aligned}
$$

Eliminating z from all equations except the third (try it!) then leads to the system

$$
\begin{aligned}
x \quad\quad &= 3 \\
y \quad &= 1 \\
z &= 2
\end{aligned}
\tag{4h}
$$

In its final form, the solution to the given system of equations can be easily read off! We have $x = 3$, $y = 1$, and $z = 2$. Geometrically, the point $(3, 1, 2)$ is the intersection of the three planes described by the three equations comprising the given system.   ■

Augmented Matrices

Observe from the preceding example that the variables x, y, and z play no significant role in each step of the reduction process, except as a reminder of the position of each coefficient in the system. With the aid of **matrices,** which are rectangular arrays of numbers, we can eliminate writing the variables at each step of the reduction and thus save ourselves a great deal of work. For example, the system

$$
\begin{aligned}
2x + 4y + 6z &= 22 \\
3x + 8y + 5z &= 27 \\
-x + y + 2z &= 2
\end{aligned}
\tag{5}
$$

may be represented by the matrix

$$
\left[
\begin{array}{ccc|c}
2 & 4 & 6 & 22 \\
3 & 8 & 5 & 27 \\
-1 & 1 & 2 & 2
\end{array}
\right]
\tag{6}
$$

The augmented matrix representing System (5)

The submatrix consisting of the first three columns of Matrix (6) is called the **coefficient matrix** of System (5). The matrix itself, (6), is referred to as the **augmented matrix** of System (5) since it is obtained by joining the matrix of coefficients to the column (matrix) of constants. The vertical line separates the column of constants from the matrix of coefficients.

The next example shows how much work you can save by using matrices instead of the standard representation of the systems of linear equations.

EXAMPLE 2 Write the augmented matrix corresponding to each equivalent system given in (4a) through (4h).

Solution   The required sequence of augmented matrices follows.

Equivalent System	Augmented Matrix

a.
$$
\begin{aligned}
2x + 4y + 6z &= 22 \\
3x + 8y + 5z &= 27 \\
-x + y + 2z &= 2
\end{aligned}
\qquad
\left[
\begin{array}{ccc|c}
2 & 4 & 6 & 22 \\
3 & 8 & 5 & 27 \\
-1 & 1 & 2 & 2
\end{array}
\right]
\tag{7a}
$$

b.
$$
\begin{aligned}
x + 2y + 3z &= 11 \\
3x + 8y + 5z &= 27 \\
-x + y + 2z &= 2
\end{aligned}
\qquad
\left[
\begin{array}{ccc|c}
1 & 2 & 3 & 11 \\
3 & 8 & 5 & 27 \\
-1 & 1 & 2 & 2
\end{array}
\right]
\tag{7b}
$$

c. $\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ -x + y + 2z &= 2 \end{aligned}$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{array}\right]$ **(7c)**

d. $\begin{aligned} x + 2y + 3z &= 11 \\ 2y - 4z &= -6 \\ 3y + 5z &= 13 \end{aligned}$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array}\right]$ **(7d)**

e. $\begin{aligned} x + 2y + 3z &= 11 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right]$ **(7e)**

f. $\begin{aligned} x + 7z &= 17 \\ y - 2z &= -3 \\ 3y + 5z &= 13 \end{aligned}$ $\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 3 & 5 & 13 \end{array}\right]$ **(7f)**

g. $\begin{aligned} x + 7z &= 17 \\ y - 2z &= -3 \\ 11z &= 22 \end{aligned}$ $\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 11 & 22 \end{array}\right]$ **(7g)**

h. $\begin{aligned} x &= 3 \\ y &= 1 \\ z &= 2 \end{aligned}$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ **(7h)**   ■

The augmented matrix in (7h) is an example of a matrix in row-reduced form. In general, an augmented matrix with m rows and n columns (called an $m \times n$ matrix) is in **row-reduced form** if it satisfies the following conditions.

> **Row-Reduced Form of a Matrix**
> **1.** Each row consisting entirely of zeros lies below all rows having nonzero entries.
> **2.** The first nonzero entry in each (nonzero) row is 1 (called a **leading 1**).
> **3.** In any two successive (nonzero) rows, the leading 1 in the lower row lies to the right of the leading 1 in the upper row.
> **4.** If a column in the coefficient matrix contains a leading 1, then the other entries in that column are zeros.

VIDEO

EXAMPLE 3 Determine which of the following matrices are in row-reduced form. If a matrix is not in row-reduced form, state the condition that is violated.

a. $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ **c.** $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

d. $\left[\begin{array}{ccc|c} 0 & 1 & 2 & -2 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$ **e.** $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 1 \end{array}\right]$ **f.** $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

g. $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \end{array}\right]$

Solution   The matrices in parts (a)–(c) are in row-reduced form.

d. This matrix is not in row-reduced form. Conditions 3 and 4 are violated: The leading 1 in row 2 lies to the left of the leading 1 in row 1. Also, column 3 contains a leading 1 in row 3 and a nonzero element above it.

e. This matrix is not in row-reduced form. Conditions 2 and 4 are violated: The first nonzero entry in row 3 is a 2, not a 1. Also, column 3 contains a leading 1 and has a nonzero entry below it.

f. This matrix is not in row-reduced form. Condition 2 is violated: The first nonzero entry in row 2 is not a leading 1.

g. This matrix is not in row-reduced form. Condition 1 is violated: Row 1 consists of all zeros and does not lie below the nonzero rows.   ■

The foregoing discussion suggests the following adaptation of the Gauss–Jordan elimination method in solving systems of linear equations using matrices. First, the three operations on the equations of a system (see page 76) translate into the following **row operations** on the corresponding augmented matrices.

> ## Row Operations
> **1.** Interchange any two rows.
> **2.** Replace any row by a nonzero constant multiple of itself.
> **3.** Replace any row by the sum of that row and a constant multiple of any other row.

We obtained the augmented matrices in Example 2 by using the same operations that we used on the equivalent system of equations in Example 1.

To help us describe the Gauss–Jordan elimination method using matrices, let's introduce some terminology. We begin by defining what is meant by a **unit column.**

> ## Unit Column
> A column in a coefficient matrix is called a **unit column** if one of the entries in the column is a 1 and the other entries are zeros.

For example, in the coefficient matrix of (7d), only the first column is in unit form; in the coefficient matrix of (7h), all three columns are in unit form. Now, the sequence of row operations that transforms the augmented matrix (7a) into the equivalent matrix (7d) in which the first column

$$2$$
$$3$$
$$-1$$

of (7a) is transformed into the unit column

$$1$$
$$0$$
$$0$$

is called **pivoting** the matrix about the element (number) 2. Similarly, we have pivoted about the element 2 in the second column of (7d), shown circled,

$$2$$
$$②$$
$$3$$

in order to obtain the augmented matrix (7g). Finally, pivoting about the element 11 in column 3 of (7g)

$$7$$
$$-2$$
$$⑪$$

leads to the augmented matrix (7h), in which all columns to the left of the vertical line are in unit form. The element about which a matrix is pivoted is called the *pivot element*.

Before looking at the next example, let's introduce the following notation for the three types of row operations.

Notation for Row Operations

Letting R_i denote the ith row of a matrix, we write:

Operation 1 $R_i \leftrightarrow R_j$ to mean: Interchange row i with row j.

Operation 2 cR_i to mean: Replace row i with c times row i.

Operation 3 $R_i + aR_j$ to mean: Replace row i with the sum of row i and a times row j.

EXAMPLE 4 Pivot the matrix about the circled element.

$$\begin{bmatrix} ③ & 5 & | & 9 \\ 2 & 3 & | & 5 \end{bmatrix}$$

Solution   Using the notation just introduced, we obtain

$$\begin{bmatrix} 3 & 5 & | & 9 \\ 2 & 3 & | & 5 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & \frac{5}{3} & | & 3 \\ 2 & 3 & | & 5 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & \frac{5}{3} & | & 3 \\ 0 & -\frac{1}{3} & | & -1 \end{bmatrix}$$

The first column, which originally contained the entry 3, is now in unit form, with a 1 where the pivot element used to be, and we are done.

Alternate Solution   In the first solution, we used operation 2 to obtain a 1 where the pivot element was originally. Alternatively, we can use operation 3 as follows:

$$\begin{bmatrix} 3 & 5 & | & 9 \\ 2 & 3 & | & 5 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 2 & | & 4 \\ 2 & 3 & | & 5 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & | & 4 \\ 0 & -1 & | & -3 \end{bmatrix}$$ ■

Note   In Example 4, the two matrices

$$\begin{bmatrix} 1 & \frac{5}{3} & | & 3 \\ 0 & -\frac{1}{3} & | & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & | & 4 \\ 0 & -1 & | & -3 \end{bmatrix}$$

look quite different, but they are in fact equivalent. You can verify this by observing that they represent the systems of equations

$$x + \frac{5}{3}y = 3 \qquad\qquad x + 2y = 4$$

$$\text{and}$$

$$-\frac{1}{3}y = -1 \qquad\qquad -y = -3$$

respectively, and both have the same solution: $x = -2$ and $y = 3$. Example 4 also shows that we can sometimes avoid working with fractions by using an appropriate row operation. ■

A summary of the Gauss–Jordan method follows.

The Gauss–Jordan Elimination Method

1. Write the augmented matrix corresponding to the linear system.
2. Interchange rows (operation 1), if necessary, to obtain an augmented matrix in which the first entry in the first row is nonzero. Then pivot the matrix about this entry.
3. Interchange the second row with any row below it, if necessary, to obtain an augmented matrix in which the second entry in the second row is nonzero. Pivot the matrix about this entry.
4. Continue until the final matrix is in row-reduced form.

Before writing the augmented matrix, be sure to write all equations with the variables on the left and constant terms on the right of the equal sign. Also, make sure that the variables are in the same order in all equations.

EXAMPLE 5 Solve the system of linear equations given by

$$3x - 2y + 8z = 9$$
$$-2x + 2y + z = 3 \qquad\qquad (8)$$
$$x + 2y - 3z = 8$$

Solution   Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{ccc|c} ③ & -2 & 8 & 9 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right]$$

$$\xrightarrow[\substack{R_3 - R_1}]{R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 2 & 19 & 27 \\ 0 & 2 & -12 & -4 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & ② & -12 & -4 \\ 0 & 2 & 19 & 27 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 2 & 19 & 27 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & \text{(31)} & 31 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{31}R_3} \begin{bmatrix} 1 & 0 & 9 & 12 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 + 6R_3]{R_1 - 9R_3} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The solution to System (8) is given by $x = 3$, $y = 4$, and $z = 1$. This may be verified by substitution into System (8) as follows:

$$3(3) - 2(4) + 8(1) = 9 \quad \checkmark$$
$$-2(3) + 2(4) + 1 \quad = 3 \quad \checkmark$$
$$3 \quad + 2(4) - 3(1) = 8 \quad \checkmark \qquad \blacksquare$$

When searching for an element to serve as a pivot, it is important to keep in mind that you may work only with the row containing the potential pivot or any row *below* it. To see what can go wrong if this caution is not heeded, consider the following augmented matrix for some linear system:

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

Observe that column 1 is in unit form. The next step in the Gauss–Jordan elimination procedure calls for obtaining a nonzero element in the second position of row 2. If you use row 1 (which is *above* the row under consideration) to help you obtain the pivot, you might proceed as follows:

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 0 & 0 & 3 & 1 \\ 1 & 1 & 2 & 3 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

As you can see, not only have we obtained a nonzero element to serve as the next pivot, but it is already a 1, thus obviating the next step. This seems like a good move. But beware, we have undone some of our earlier work: Column 1 is no longer a unit column where a 1 appears first. The correct move in this case is to interchange row 2 with row 3 in the first augmented matrix.

The next example illustrates how to handle a situation in which the first entry in row 1 of the augmented matrix is zero.

Explore & Discuss

1. Can the phrase "a nonzero constant multiple of itself" in a type-2 row operation be replaced by "a constant multiple of itself"? Explain.

2. Can a row of an augmented matrix be replaced by a row obtained by adding a constant to every element in that row without changing the solution of the system of linear equations? Explain.

EXAMPLE 6 Solve the system of linear equations given by

$$2y + 3z = 7$$
$$3x + 6y - 12z = -3$$
$$5x - 2y + 2z = -7$$

Solution Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\begin{bmatrix} 0 & 2 & 3 & | & 7 \\ 3 & 6 & -12 & | & -3 \\ 5 & -2 & 2 & | & -7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 6 & -12 & | & -3 \\ 0 & 2 & 3 & | & 7 \\ 5 & -2 & 2 & | & -7 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 2 & -4 & | & -1 \\ 0 & 2 & 3 & | & 7 \\ 5 & -2 & 2 & | & -7 \end{bmatrix}$$

$$\xrightarrow{R_3 - 5R_1} \begin{bmatrix} 1 & 2 & -4 & | & -1 \\ 0 & 2 & 3 & | & 7 \\ 0 & -12 & 22 & | & -2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -4 & | & -1 \\ 0 & 1 & \frac{3}{2} & | & \frac{7}{2} \\ 0 & -12 & 22 & | & -2 \end{bmatrix}$$

$$\xrightarrow[R_3 + 12R_2]{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -7 & | & -8 \\ 0 & 1 & \frac{3}{2} & | & \frac{7}{2} \\ 0 & 0 & 40 & | & 40 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{40}R_3} \begin{bmatrix} 1 & 0 & -7 & | & -8 \\ 0 & 1 & \frac{3}{2} & | & \frac{7}{2} \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 - \frac{3}{2}R_3]{R_1 + 7R_3} \begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

The solution to the system is given by $x = -1$, $y = 2$, and $z = 1$; this may be verified by substitution into the system. ■

APPLIED EXAMPLE 7 **Manufacturing: Production Scheduling** Complete the solution to Example 1 in Section 2.1, page 70.

Solution To complete the solution of the problem posed in Example 1, recall that the mathematical formulation of the problem led to the following system of linear equations:

$$2x + y + z = 180$$
$$x + 3y + 2z = 300$$
$$2x + y + 2z = 240$$

where x, y, and z denote the respective numbers of type-A, type-B, and type-C souvenirs to be made.

Solving the foregoing system of linear equations by the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$
\begin{bmatrix}
2 & 1 & 1 & | & 180 \\
1 & 3 & 2 & | & 300 \\
2 & 1 & 2 & | & 240
\end{bmatrix}
\xrightarrow{R_1 \leftrightarrow R_2}
\begin{bmatrix}
1 & 3 & 2 & | & 300 \\
2 & 1 & 1 & | & 180 \\
2 & 1 & 2 & | & 240
\end{bmatrix}
$$

$$
\xrightarrow[R_3 - 2R_1]{R_2 - 2R_1}
\begin{bmatrix}
1 & 3 & 2 & | & 300 \\
0 & -5 & -3 & | & -420 \\
0 & -5 & -2 & | & -360
\end{bmatrix}
$$

$$
\xrightarrow{-\frac{1}{5}R_2}
\begin{bmatrix}
1 & 3 & 2 & | & 300 \\
0 & 1 & \frac{3}{5} & | & 84 \\
0 & -5 & -2 & | & -360
\end{bmatrix}
$$

$$
\xrightarrow[R_3 + 5R_2]{R_1 - 3R_2}
\begin{bmatrix}
1 & 0 & \frac{1}{5} & | & 48 \\
0 & 1 & \frac{3}{5} & | & 84 \\
0 & 0 & 1 & | & 60
\end{bmatrix}
$$

$$
\xrightarrow[R_2 - \frac{3}{5}R_3]{R_1 - \frac{1}{5}R_3}
\begin{bmatrix}
1 & 0 & 0 & | & 36 \\
0 & 1 & 0 & | & 48 \\
0 & 0 & 1 & | & 60
\end{bmatrix}
$$

Thus, $x = 36$, $y = 48$, and $z = 60$; that is, Ace Novelty should make 36 type-A souvenirs, 48 type-B souvenirs, and 60 type-C souvenirs in order to use all available machine time.

2.2 Self-Check Exercises

1. Solve the system of linear equations

$$
\begin{aligned}
2x + 3y + z &= 6 \\
x - 2y + 3z &= -3 \\
3x + 2y - 4z &= 12
\end{aligned}
$$

using the Gauss–Jordan elimination method.

2. A farmer has 200 acres of land suitable for cultivating crops A, B, and C. The cost per acre of cultivating crop A, crop B, and crop C is $40, $60, and $80, respectively. The farmer has $12,600 available for land cultivation. Each acre of crop A requires 20 labor-hours, each acre of crop B requires 25 labor-hours, and each acre of crop C requires 40 labor-hours. The farmer has a maximum of 5950 labor-hours available. If she wishes to use all of her cultivatable land, the entire budget, and all the labor available, how many acres of each crop should she plant?

Solutions to Self-Check Exercises 2.2 can be found on page 89.

2.2 Concept Questions

1. a. Explain what it means for two systems of linear equations to be equivalent to each other.
  b. Give the meaning of the following notation used for row operations in the Gauss–Jordan elimination method:

  i. $R_i \leftrightarrow R_j$ **ii.** cR_i **iii.** $R_i + aR_j$

2. a. What is an augmented matrix? A coefficient matrix? A unit column?
  b. Explain what is meant by a pivot operation.

3. Suppose that a matrix is in row-reduced form.
  a. What is the position of a row consisting entirely of zeros relative to the nonzero rows?
  b. What is the first nonzero entry in each row?
  c. What is the position of the leading 1s in successive nonzero rows?
  d. If a column contains a leading 1, then what is the value of the other entries in that column?

2.2   Exercises

In Exercises 1–4, write the augmented matrix corresponding to each system of equations.

1. $2x - 3y = 7$
$3x + y = 4$

2. $3x + 7y - 8z = 5$
$x + 3z = -2$
$4x - 3y = 7$

3. $-y + 2z = 6$
$2x + 2y - 8z = 7$
$3y + 4z = 0$

4. $3x_1 + 2x_2 = 0$
$x_1 - x_2 + 2x_3 = 4$
$2x_2 - 3x_3 = 5$

In Exercises 5–8, write the system of equations corresponding to each augmented matrix.

5. $\begin{bmatrix} 3 & 2 & | & -4 \\ 1 & -1 & | & 5 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 3 & 2 & | & 4 \\ 1 & -1 & -2 & | & -3 \\ 4 & 0 & 3 & | & 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 & 2 & | & 4 \\ 2 & 0 & 0 & | & 5 \\ 3 & -3 & 2 & | & 6 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 3 & 1 & | & 6 \\ 4 & 3 & 2 & | & 5 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

In Exercises 9–18, indicate whether the matrix is in row-reduced form.

9. $\begin{bmatrix} 1 & 0 & | & 3 \\ 0 & 1 & | & -2 \end{bmatrix}$

10. $\begin{bmatrix} 1 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix}$

11. $\begin{bmatrix} 0 & 1 & | & 3 \\ 1 & 0 & | & 5 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 1 & | & 3 \\ 0 & 0 & | & 5 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 5 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 2 & | & -3 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 & 1 & | & 3 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & -1 & | & 6 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & | & -10 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$

17. $\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 0 & 1 & 2 & | & 4 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 6 \\ 0 & 0 & 0 & | & 4 \\ 0 & 0 & 1 & | & 5 \end{bmatrix}$

In Exercises 19–26, pivot the system about the circled element.

19. $\begin{bmatrix} ②\ 4 & | & 8 \\ 3 & 1 & | & 2 \end{bmatrix}$

20. $\begin{bmatrix} 3 & 2 & | & 6 \\ ④\ 2 & | & 5 \end{bmatrix}$

21. $\begin{bmatrix} \boxed{-1} & 2 & | & 3 \\ 6 & 4 & | & 2 \end{bmatrix}$

22. $\begin{bmatrix} ①\ 3 & | & 4 \\ 2 & 4 & | & 6 \end{bmatrix}$

23. $\begin{bmatrix} ② & 4 & 6 & | & 12 \\ 2 & 3 & 1 & | & 5 \\ 3 & -1 & 2 & | & 4 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 3 & 2 & | & 4 \\ ② & 4 & 8 & | & 6 \\ -1 & 2 & 3 & | & 4 \end{bmatrix}$

25. $\begin{bmatrix} 0 & 1 & 3 & | & 4 \\ 2 & 4 & ① & | & 3 \\ 5 & 6 & 2 & | & -4 \end{bmatrix}$

26. $\begin{bmatrix} 1 & 2 & 3 & | & 5 \\ 0 & \boxed{-3} & 3 & | & 2 \\ 0 & 4 & -1 & | & 3 \end{bmatrix}$

In Exercises 27–30, fill in the missing entries by performing the indicated row operations to obtain the row-reduced matrices.

27. $\begin{bmatrix} 3 & 9 & | & 6 \\ 2 & 1 & | & 4 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} \cdot & \cdot & | & \cdot \\ 2 & 1 & | & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1}$

$\begin{bmatrix} 1 & 3 & | & 2 \\ \cdot & \cdot & | & \cdot \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2} \begin{bmatrix} 1 & 3 & | & 2 \\ \cdot & \cdot & | & \cdot \end{bmatrix} \xrightarrow{R_1 - 3R_2} \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 0 \end{bmatrix}$

28. $\begin{bmatrix} 1 & 2 & | & 1 \\ 2 & 3 & | & -1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & | & 1 \\ \cdot & \cdot & | & \cdot \end{bmatrix} \xrightarrow{-R_2}$

$\begin{bmatrix} 1 & 2 & | & 1 \\ \cdot & \cdot & | & \cdot \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & | & -5 \\ 0 & 1 & | & 3 \end{bmatrix}$

29. $\begin{bmatrix} 1 & 3 & 1 & | & 3 \\ 3 & 8 & 3 & | & 7 \\ 2 & -3 & 1 & | & -10 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 3 & 1 & | & 3 \\ \cdot & \cdot & \cdot & | & \cdot \\ \cdot & \cdot & \cdot & | & \cdot \end{bmatrix} \xrightarrow{-R_2}$

$\begin{bmatrix} 1 & 3 & 1 & | & 3 \\ \cdot & \cdot & \cdot & | & \cdot \\ 0 & -9 & -1 & | & -16 \end{bmatrix} \xrightarrow[R_3 + 9R_2]{R_1 - 3R_2}$

$\begin{bmatrix} \cdot & \cdot & \cdot & | & \cdot \\ 0 & 1 & 0 & | & 2 \\ \cdot & \cdot & \cdot & | & \cdot \end{bmatrix} \xrightarrow[-R_3]{R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$

30. $\begin{bmatrix} 0 & 1 & 3 & | & -4 \\ 1 & 2 & 1 & | & 7 \\ 1 & -2 & 0 & | & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \cdot & \cdot & \cdot & | & \cdot \\ \cdot & \cdot & \cdot & | & \cdot \\ 1 & -2 & 0 & | & 1 \end{bmatrix}$

$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 & 1 & | & 7 \\ 0 & 1 & 3 & | & -4 \\ \cdot & \cdot & \cdot & | & \cdot \end{bmatrix} \xrightarrow[R_3 + 4R_2]{R_1 + \frac{1}{2}R_3} \begin{bmatrix} \cdot & \cdot & \cdot & | & \cdot \\ 0 & 1 & 3 & | & -4 \\ \cdot & \cdot & \cdot & | & \cdot \end{bmatrix}$

$\xrightarrow{\frac{1}{11}R_3} \begin{bmatrix} 1 & 0 & \frac{1}{2} & | & 4 \\ 0 & 1 & 3 & | & -4 \\ \cdot & \cdot & \cdot & | & \cdot \end{bmatrix} \xrightarrow[R_2 - 3R_3]{R_1 - \frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 & | & 5 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$

31. Write a system of linear equations for the augmented matrix of Exercise 27. Using the results of Exercise 27, determine the solution of the system.

32. Repeat Exercise 31 for the augmented matrix of Exercise 28.

33. Repeat Exercise 31 for the augmented matrix of Exercise 29.

34. Repeat Exercise 31 for the augmented matrix of Exercise 30.

In Exercises 35–50, solve the system of linear equations using the Gauss–Jordan elimination method.

35.
$$x - 2y = 8$$
$$3x + 4y = 4$$

36.
$$3x + y = 1$$
$$-7x - 2y = -1$$

37.
$$2x - 3y = -8$$
$$4x + y = -2$$

38.
$$5x + 3y = 9$$
$$-2x + y = -8$$

39.
$$x + y + z = 0$$
$$2x - y + z = 1$$
$$x + y - 2z = 2$$

40.
$$2x + y - 2z = 4$$
$$x + 3y - z = -3$$
$$3x + 4y - z = 7$$

41.
$$2x + 2y + z = 9$$
$$x + z = 4$$
$$4y - 3z = 17$$

42.
$$2x + 3y - 2z = 10$$
$$3x - 2y + 2z = 0$$
$$4x - y + 3z = -1$$

43.
$$- x_2 + x_3 = 2$$
$$4x_1 - 3x_2 + 2x_3 = 16$$
$$3x_1 + 2x_2 + x_3 = 11$$

44.
$$2x + 4y - 6z = 38$$
$$x + 2y + 3z = 7$$
$$3x - 4y + 4z = -19$$

45.
$$x_1 - 2x_2 + x_3 = 6$$
$$2x_1 + x_2 - 3x_3 = -3$$
$$x_1 - 3x_2 + 3x_3 = 10$$

46.
$$2x + 3y - 6z = -11$$
$$x - 2y + 3z = 9$$
$$3x + y = 7$$

47.
$$2x + 3z = -1$$
$$3x - 2y + z = 9$$
$$x + y + 4z = 4$$

48.
$$2x_1 - x_2 + 3x_3 = -4$$
$$x_1 - 2x_2 + x_3 = -1$$
$$x_1 - 5x_2 + 2x_3 = -3$$

49.
$$x_1 - x_2 + 3x_3 = 14$$
$$x_1 + x_2 + x_3 = 6$$
$$-2x_1 - x_2 + x_3 = -4$$

50.
$$2x_1 - x_2 - x_3 = 0$$
$$3x_1 + 2x_2 + x_3 = 7$$
$$x_1 + 2x_2 + 2x_3 = 5$$

The problems in Exercises 51–63 correspond to those in Exercises 15–27, Section 2.1. Use the results of your previous work to help you solve these problems.

51. AGRICULTURE The Johnson Farm has 500 acres of land allotted for cultivating corn and wheat. The cost of cultivating corn and wheat (including seeds and labor) is $42 and $30 per acre, respectively. Jacob Johnson has $18,600 available for cultivating these crops. If he wishes to use all the allotted land and his entire budget for cultivating these two crops, how many acres of each crop should he plant?

52. INVESTMENTS Michael Perez has a total of $2000 on deposit with two savings institutions. One pays interest at the rate of 6%/year, whereas the other pays interest at the rate of 8%/year. If Michael earned a total of $144 in interest during a single year, how much does he have on deposit in each institution?

53. MIXTURES The Coffee Shoppe sells a coffee blend made from two coffees, one costing $5/lb and the other costing $6/lb. If the blended coffee sells for $5.60/lb, find how much of each coffee is used to obtain the desired blend. Assume that the weight of the blended coffee is 100 lb.

54. INVESTMENTS Kelly Fisher has a total of $30,000 invested in two municipal bonds that have yields of 8% and 10% interest per year, respectively. If the interest Kelly receives from the bonds in a year is $2640, how much does she have invested in each bond?

55. RIDERSHIP The total number of passengers riding a certain city bus during the morning shift is 1000. If the child's fare is $.50, the adult fare is $1.50, and the total revenue from the fares in the morning shift is $1300, how many children and how many adults rode the bus during the morning shift?

56. REAL ESTATE Cantwell Associates, a real estate developer, is planning to build a new apartment complex consisting of one-bedroom units and two- and three-bedroom townhouses. A total of 192 units is planned, and the number of family units (two- and three-bedroom townhouses) will equal the number of one-bedroom units. If the number of one-bedroom units will be 3 times the number of three-bedroom units, find how many units of each type will be in the complex.

57. INVESTMENT PLANNING The annual returns on Sid Carrington's three investments amounted to $21,600: 6% on a savings account, 8% on mutual funds, and 12% on bonds. The amount of Sid's investment in bonds was twice the amount of his investment in the savings account, and the interest earned from his investment in bonds was equal to the dividends he received from his investment in mutual funds. Find how much money he placed in each type of investment.

58. INVESTMENT CLUB A private investment club has $200,000 earmarked for investment in stocks. To arrive at an acceptable overall level of risk, the stocks that management is considering have been classified into three categories: high risk, medium risk, and low risk. Management estimates that high-risk stocks will have a rate of return of 15%/year; medium-risk stocks, 10%/year; and low-risk stocks, 6%/year. The members have decided that the investment in low-risk stocks should be equal to the sum of the investments in the stocks of the other two categories. Determine how much the club should invest in each type of stock if the investment goal is to have a return of $20,000/year on the total investment. (Assume that all the money available for investment is invested.)

59. MIXTURE PROBLEM—FERTILIZER Lawnco produces three grades of commercial fertilizers. A 100-lb bag of grade-A fertilizer contains 18 lb of nitrogen, 4 lb of phosphate, and 5 lb of potassium. A 100-lb bag of grade-B fertilizer contains 20 lb of nitrogen and 4 lb each of phosphate and potassium. A 100-lb bag of grade-C fertilizer contains 24 lb of nitrogen, 3 lb of phosphate, and 6 lb of potassium. How many 100-lb bags of each of the three grades of fertilizers should Lawnco produce if 26,400 lb of nitrogen, 4900 lb of phosphate, and 6200 lb of potassium are available and all the nutrients are used?

60. BOX-OFFICE RECEIPTS A theater has a seating capacity of 900 and charges $4 for children, $6 for students, and $8 for adults. At a certain screening with full attendance, there were half as many adults as children and students combined. The receipts totaled $5600. How many children attended the show?

61. MANAGEMENT DECISIONS The management of Hartman Rent-A-Car has allocated $1.5 million to buy a fleet of new automobiles consisting of compact, intermediate-size, and full-size cars. Compacts cost $12,000 each, intermediate-size cars cost $18,000 each, and full-size cars cost $24,000 each. If Hartman purchases twice as many compacts as intermediate-size cars and the total number of cars to be purchased is 100, determine how many cars of each type will be purchased. (Assume that the entire budget will be used.)

62. INVESTMENT CLUBS The management of a private investment club has a fund of $200,000 earmarked for investment in stocks. To arrive at an acceptable overall level of risk, the stocks that management is considering have been classified into three categories: high risk, medium risk, and low risk. Management estimates that high-risk stocks will have a rate of return of 15%/year; medium-risk stocks, 10%/year; and low-risk stocks, 6%/year. The investment in low-risk stocks is to be twice the sum of the investments in stocks of the other two categories. If the investment goal is to have an average rate of return of 9%/year on the total investment, determine how much the club should invest in each type of stock. (Assume that all of the money available for investment is invested.)

63. DIET PLANNING A dietitian wishes to plan a meal around three foods. The percent of the daily requirements of proteins, carbohydrates, and iron contained in each ounce of the three foods is summarized in the following table:

	Food I	Food II	Food III
Proteins (%)	10	6	8
Carbohydrates (%)	10	12	6
Iron (%)	5	4	12

Determine how many ounces of each food the dietitian should include in the meal to meet exactly the daily requirement of proteins, carbohydrates, and iron (100% of each).

64. INVESTMENTS Mr. and Mrs. Garcia have a total of $100,000 to be invested in stocks, bonds, and a money market account. The stocks have a rate of return of 12%/year, while the bonds and the money market account pay 8%/year and 4%/year, respectively. The Garcias have stipulated that the amount invested in the money market account should be equal to the sum of 20% of the amount invested in stocks and 10% of the amount invested in bonds. How should the Garcias allocate their resources if they require an annual income of $10,000 from their investments?

65. BOX-OFFICE RECEIPTS For the opening night at the Opera House, a total of 1000 tickets were sold. Front orchestra seats cost $80 apiece, rear orchestra seats cost $60 apiece, and front balcony seats cost $50 apiece. The combined number of tickets sold for the front orchestra and rear orchestra exceeded twice the number of front balcony tickets sold by 400. The total receipts for the performance were $62,800. Determine how many tickets of each type were sold.

66. PRODUCTION SCHEDULING A manufacturer of women's blouses makes three types of blouses: sleeveless, short-sleeve, and long-sleeve. The time (in minutes) required by each department to produce a dozen blouses of each type is shown in the following table:

	Sleeveless	Short-Sleeve	Long-Sleeve
Cutting	9	12	15
Sewing	22	24	28
Packaging	6	8	8

The cutting, sewing, and packaging departments have available a maximum of 80, 160, and 48 labor-hours, respectively, per day. How many dozens of each type of blouse can be produced each day if the plant is operated at full capacity?

67. BUSINESS TRAVEL EXPENSES An executive of Trident Communications recently traveled to London, Paris, and Rome. He paid $180, $230, and $160 per night for lodging in London, Paris, and Rome, respectively, and his hotel bills totaled $2660. He spent $110, $120, and $90 per day for his meals in London, Paris, and Rome, respectively, and his expenses for meals totaled $1520. If he spent as many days in London as he did in Paris and Rome combined, how many days did he stay in each city?

68. VACATION COSTS Joan and Dick spent 2 wk (14 nights) touring four cities on the East Coast—Boston, New York, Philadelphia, and Washington. They paid $120, $200, $80, and $100 per night for lodging in each city, respectively, and their total hotel bill came to $2020. The number of days they spent in New York was the same as the total number of days they spent in Boston and Washington, and the couple spent 3 times as many days in New York as they did in Philadelphia. How many days did Joan and Dick stay in each city?

In Exercises 69 and 70, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

69. An equivalent system of linear equations can be obtained from a system of equations by replacing one of its equations by any constant multiple of itself.

70. If the augmented matrix corresponding to a system of three linear equations in three variables has a row of the form $[0 \quad 0 \quad 0 \mid a]$, where a is a nonzero number, then the system has no solution.

2.2 Solutions to Self-Check Exercises

1. We obtain the following sequence of equivalent augmented matrices:

$$\begin{bmatrix} 2 & 3 & 1 & | & 6 \\ 1 & -2 & 3 & | & -3 \\ 3 & 2 & -4 & | & 12 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} \textcircled{1} & -2 & 3 & | & -3 \\ 2 & 3 & 1 & | & 6 \\ 3 & 2 & -4 & | & 12 \end{bmatrix}$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 3 & | & -3 \\ 0 & 7 & -5 & | & 12 \\ 0 & 8 & -13 & | & 21 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & -2 & 3 & | & -3 \\ 0 & \textcircled{8} & -13 & | & 21 \\ 0 & 7 & -5 & | & 12 \end{bmatrix} \xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & -2 & 3 & | & -3 \\ 0 & 1 & -8 & | & 9 \\ 0 & 7 & -5 & | & 12 \end{bmatrix}$$

$$\xrightarrow[R_3 - 7R_2]{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -13 & | & 15 \\ 0 & 1 & -8 & | & 9 \\ 0 & 0 & 51 & | & -51 \end{bmatrix} \xrightarrow{\frac{1}{51}R_3} \begin{bmatrix} 1 & 0 & -13 & | & 15 \\ 0 & 1 & -8 & | & 9 \\ 0 & 0 & \textcircled{1} & | & -1 \end{bmatrix}$$

$$\xrightarrow[R_2 + 8R_3]{R_1 + 13R_3} \begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

The solution to the system is $x = 2$, $y = 1$, and $z = -1$.

2. Referring to the solution of Exercise 2, Self-Check Exercises 2.1, we see that the problem reduces to solving the following system of linear equations:

$$\begin{aligned} x + y + z &= 200 \\ 40x + 60y + 80z &= 12,600 \\ 20x + 25y + 40z &= 5,950 \end{aligned}$$

Using the Gauss–Jordan elimination method, we have

$$\begin{bmatrix} \textcircled{1} & 1 & 1 & | & 200 \\ 40 & 60 & 80 & | & 12,600 \\ 20 & 25 & 40 & | & 5,950 \end{bmatrix} \xrightarrow[R_3 - 20R_1]{R_2 - 40R_1} \begin{bmatrix} 1 & 1 & 1 & | & 200 \\ 0 & \textcircled{20} & 40 & | & 4600 \\ 0 & 5 & 20 & | & 1950 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{20}R_2} \begin{bmatrix} 1 & 1 & 1 & | & 200 \\ 0 & 1 & 2 & | & 230 \\ 0 & 5 & 20 & | & 1950 \end{bmatrix} \xrightarrow[R_3 - 5R_2]{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 & | & -30 \\ 0 & 1 & 2 & | & 230 \\ 0 & 0 & \textcircled{10} & | & 800 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{10}R_3} \begin{bmatrix} 1 & 0 & -1 & | & -30 \\ 0 & 1 & 2 & | & 230 \\ 0 & 0 & 1 & | & 80 \end{bmatrix} \xrightarrow[R - 2R_3]{R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 & | & 50 \\ 0 & 1 & 0 & | & 70 \\ 0 & 0 & 1 & | & 80 \end{bmatrix}$$

From the last augmented matrix in reduced form, we see that $x = 50$, $y = 70$, and $z = 80$. Therefore, the farmer should plant 50 acres of crop A, 70 acres of crop B, and 80 acres of crop C.

USING TECHNOLOGY

Systems of Linear Equations: Unique Solutions

Solving a System of Linear Equations Using the Gauss–Jordan Method

The three matrix operations can be performed on a matrix using a graphing utility. The commands are summarized in the following table.

	Calculator Function		
Operation	TI-83/84	TI-86	
$R_i \leftrightarrow R_j$	**rowSwap**([A], i, j)	**rSwap**(A, i, j)	or equivalent
cR_i	***row**(c, [A], i)	**multR**(c, A, i)	or equivalent
$R_i + aR_j$	***row+**(a, [A], j, i)	**mRAdd**(a, A, j, i)	or equivalent

When a row operation is performed on a matrix, the result is stored as an answer in the calculator. If another operation is performed on this matrix, then the matrix is erased. Should a mistake be made in the operation, the previous matrix may be lost. For this reason, you should store the results of each operation. We do this by pressing **sto**, followed by the name of a matrix, and then **enter**. We use this process in the following example.

EXAMPLE 1 Use a graphing utility to solve the following system of linear equations by the Gauss–Jordan method (see Example 5 in Section 2.2):

$$\begin{aligned} 3x - 2y + 8z &= 9 \\ -2x + 2y + z &= 3 \\ x + 2y - 3z &= 8 \end{aligned}$$

Solution   Using the Gauss–Jordan method, we obtain the following sequence of equivalent matrices.

$$\begin{bmatrix} 3 & -2 & 8 & \bigm| & 9 \\ -2 & 2 & 1 & \bigm| & 3 \\ 1 & 2 & -3 & \bigm| & 8 \end{bmatrix} \xrightarrow{\text{*row+ } (1, [A], 2, 1) \blacktriangleright B}$$

$$\begin{bmatrix} 1 & 0 & 9 & \bigm| & 12 \\ -2 & 2 & 1 & \bigm| & 3 \\ 1 & 2 & -3 & \bigm| & 8 \end{bmatrix} \xrightarrow{\text{*row+ } (2, [B], 1, 2) \blacktriangleright C}$$

$$\begin{bmatrix} 1 & 0 & 9 & \bigm| & 12 \\ 0 & 2 & 19 & \bigm| & 27 \\ 1 & 2 & -3 & \bigm| & 8 \end{bmatrix} \xrightarrow{\text{*row+ } (-1, [C], 1, 3) \blacktriangleright B}$$

$$\begin{bmatrix} 1 & 0 & 9 & \bigm| & 12 \\ 0 & 2 & 19 & \bigm| & 27 \\ 0 & 2 & -12 & \bigm| & -4 \end{bmatrix} \xrightarrow{\text{*row}(\frac{1}{2}, [B], 2) \blacktriangleright C}$$

$$\begin{bmatrix} 1 & 0 & 9 & \bigm| & 12 \\ 0 & 1 & 9.5 & \bigm| & 13.5 \\ 0 & 2 & -12 & \bigm| & -4 \end{bmatrix} \xrightarrow{\text{*row+ } (-2, [C], 2, 3) \blacktriangleright B}$$

$$\begin{bmatrix} 1 & 0 & 9 & \bigm| & 12 \\ 0 & 1 & 9.5 & \bigm| & 13.5 \\ 0 & 0 & -31 & \bigm| & -31 \end{bmatrix} \xrightarrow{\text{*row}(-\frac{1}{31}, [B], 3) \blacktriangleright C}$$

$$\begin{bmatrix} 1 & 0 & 9 & \bigm| & 12 \\ 0 & 1 & 9.5 & \bigm| & 13.5 \\ 0 & 0 & 1 & \bigm| & 1 \end{bmatrix} \xrightarrow{\text{*row+ } (-9, [C], 3, 1) \blacktriangleright B}$$

$$\begin{bmatrix} 1 & 0 & 0 & \bigm| & 3 \\ 0 & 1 & 9.5 & \bigm| & 13.5 \\ 0 & 0 & 1 & \bigm| & 1 \end{bmatrix} \xrightarrow{\text{*row+ } (-9.5, [B], 3, 2) \blacktriangleright C} \begin{bmatrix} 1 & 0 & 0 & \bigm| & 3 \\ 0 & 1 & 0 & \bigm| & 4 \\ 0 & 0 & 1 & \bigm| & 1 \end{bmatrix}$$

The last matrix is in row-reduced form, and we see that the solution of the system is $x = 3$, $y = 4$, and $z = 1$.   ■

Using rref (TI-83/84 and TI-86) to Solve a System of Linear Equations

The operation **rref** (or equivalent function in your utility, if there is one) will transform an augmented matrix into one that is in row-reduced form. For example, using **rref**, we find

$$\begin{bmatrix} 3 & -2 & 8 & \bigm| & 9 \\ -2 & 2 & 1 & \bigm| & 3 \\ 1 & 2 & -3 & \bigm| & 8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & \bigm| & 3 \\ 0 & 1 & 0 & \bigm| & 4 \\ 0 & 0 & 1 & \bigm| & 1 \end{bmatrix}$$

as obtained earlier!

Using SIMULT (TI-86) to Solve a System of Equations

The operation **SIMULT** (or equivalent operation on your utility, if there is one) of a graphing utility can be used to solve a system of n linear equations in n variables, where n is an integer between 2 and 30, inclusive.

EXAMPLE 2 Use the SIMULT operation to solve the system of Example 1.

Solution Call for the SIMULT operation. Since the system under consideration has three equations in three variables, enter $n = 3$. Next, enter a1, 1 = 3, a1, 2 = -2, a1, 3 = 8, b1 = 9, a2, 1 = -2, . . . , b3 = 8. Select <SOLVE> and the display

$$x1 = 3$$
$$x2 = 4$$
$$x3 = 1$$

appears on the screen, giving $x = 3$, $y = 4$, and $z = 1$ as the required solution.   ∎

TECHNOLOGY EXERCISES

Use a graphing utility to solve the system of equations (a) by the Gauss–Jordan method, (b) using the rref operation, and (c) using SIMULT.

1. $x_1 - 2x_2 + 2x_3 - 3x_4 = -7$
   $3x_1 + 2x_2 - x_3 + 5x_4 = 22$
   $2x_1 - 3x_2 + 4x_3 - x_4 = -3$
   $3x_1 - 2x_2 - x_3 + 2x_4 = 12$

2. $2x_1 - x_2 + 3x_3 - 2x_4 = -2$
   $x_1 - 2x_2 + x_3 - 3x_4 = 2$
   $x_1 - 5x_2 + 2x_3 + 3x_4 = -6$
   $-3x_1 + 3x_2 - 4x_3 - 4x_4 = 9$

3. $2x_1 + x_2 + 3x_3 - x_4 = 9$
   $-x_1 - 2x_2 \qquad - 3x_4 = -1$
   $x_1 \qquad - 3x_3 + x_4 = 10$
   $x_1 - x_2 - x_3 - x_4 = 8$

4. $x_1 - 2x_2 - 2x_3 + x_4 = 1$
   $2x_1 - x_2 + 2x_3 + 3x_4 = -2$
   $-x_1 - 5x_2 + 7x_3 - 2x_4 = 3$
   $3x_1 - 4x_2 + 3x_3 + 4x_4 = -4$

5. $2x_1 - 2x_2 + 3x_3 - x_4 + 2x_5 = 16$
   $3x_1 + x_2 - 2x_3 + x_4 - 3x_5 = -11$
   $x_1 + 3x_2 - 4x_3 + 3x_4 - x_5 = -13$
   $2x_1 - x_2 + 3x_3 - 2x_4 + 2x_5 = 15$
   $3x_1 + 4x_2 - 3x_3 + 5x_4 - x_5 = -10$

6. $2.1x_1 - 3.2x_2 + 6.4x_3 + 7x_4 - 3.2x_5 = 54.3$
   $4.1x_1 + 2.2x_2 - 3.1x_3 - 4.2x_4 + 3.3x_5 = -20.81$
   $3.4x_1 - 6.2x_2 + 4.7x_3 + 2.1x_4 - 5.3x_5 = 24.7$
   $4.1x_1 + 7.3x_2 + 5.2x_3 + 6.1x_4 - 8.2x_5 = 29.25$
   $2.8x_1 + 5.2x_2 + 3.1x_3 + 5.4x_4 + 3.8x_5 = 43.72$

2.3 Systems of Linear Equations: Underdetermined and Overdetermined Systems

In this section, we continue our study of systems of linear equations. More specifically, we look at systems that have infinitely many solutions and those that have no solution. We also study systems of linear equations in which the number of variables is not equal to the number of equations in the system.

Solution(s) of Linear Equations

Our first example illustrates the situation in which a system of linear equations has infinitely many solutions.

EXAMPLE 1 A System of Equations with an Infinite Number of Solutions
Solve the system of linear equations given by

$$x + 2y - 3z = -2$$
$$3x - y - 2z = 1 \qquad\qquad \textbf{(9)}$$
$$2x + 3y - 5z = -3$$

Solution   Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{ccc|c} ① & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & ⊖7 & 7 & 7 \\ 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow{-\frac{1}{7}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 + R_2]{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last augmented matrix is in row-reduced form. Interpreting it as a system of linear equations gives

$$\begin{aligned} x - z &= 0 \\ y - z &= -1 \end{aligned}$$

a system of two equations in the three variables x, y, and z.

Let's now single out one variable—say, z—and solve for x and y in terms of it. We obtain

$$\begin{aligned} x &= z \\ y &= z - 1 \end{aligned}$$

If we assign a particular value to z—say, $z = 0$—we obtain $x = 0$ and $y = -1$, giving the solution $(0, -1, 0)$ to System (9). By setting $z = 1$, we obtain the solution $(1, 0, 1)$. In general, if we set $z = t$, where t represents some real number (called a parameter), we obtain a solution given by $(t, t - 1, t)$. Since the parameter t may be any real number, we see that System (9) has infinitely many solutions. Geometrically, the solutions of System (9) lie on the straight line in three-dimensional space given by the intersection of the three planes determined by the three equations in the system. ∎

Note   In Example 1 we chose the parameter to be z because it is more convenient to solve for x and y (both the x- and y-columns are in unit form) in terms of z. ∎

The next example shows what happens in the elimination procedure when the system does not have a solution.

EXAMPLE 2 A System of Equations That Has No Solution Solve the system of linear equations given by

$$\begin{aligned} x + y + z &= 1 \\ 3x - y - z &= 4 \\ x + 5y + 5z &= -1 \end{aligned} \tag{10}$$

Solution   Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent augmented matrices:

$$\left[\begin{array}{ccc|c} ① & 1 & 1 & 1 \\ 3 & -1 & -1 & 4 \\ 1 & 5 & 5 & -1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 4 & 4 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Observe that row 3 in the last matrix reads $0x + 0y + 0z = -1$—that is, $0 = -1$! We therefore conclude that System (10) is inconsistent and has no solution. Geometrically, we have a situation in which two of the planes intersect in a straight line but the third plane is parallel to this line of intersection of the two planes and does not intersect it. Consequently, there is no point of intersection of the three planes. ∎

Example 2 illustrates the following more general result of using the Gauss–Jordan elimination procedure.

Systems with No Solution

If there is a row in an augmented matrix containing all zeros to the left of the vertical line and a nonzero entry to the right of the line, then the corresponding system of equations has no solution.

It may have dawned on you that in all the previous examples we have dealt only with systems involving exactly the same number of linear equations as there are variables. However, systems in which the number of equations is different from the number of variables also occur in practice. Indeed, we will consider such systems in Examples 3 and 4.

The following theorem provides us with some preliminary information on a system of linear equations.

THEOREM 1

a. If the number of equations is greater than or equal to the number of variables in a linear system, then one of the following is true:
   i. The system has no solution.
   ii. The system has exactly one solution.
   iii. The system has infinitely many solutions.
b. If there are fewer equations than variables in a linear system, then the system either has no solution or it has infinitely many solutions.

Note   Theorem 1 may be used to tell us, before we even begin to solve a problem, what the nature of the solution may be.   ∎

Although we will not prove this theorem, you should recall that we have illustrated geometrically part (a) for the case in which there are exactly as many equations (three) as there are variables. To show the validity of part (b), let us once again consider the case in which a system has three variables. Now, if there is only one equation in the system, then it is clear that there are infinitely many solutions corresponding geometrically to all the points lying on the plane represented by the equation.

Next, if there are two equations in the system, then *only* the following possibilities exist:

1. The two planes are parallel and distinct.
2. The two planes intersect in a straight line.
3. The two planes are coincident (the two equations define the same plane) (Figure 6).

(a) No solution   **(b)** Infinitely many solutions   **(c)** Infinitely many solutions

FIGURE **6**
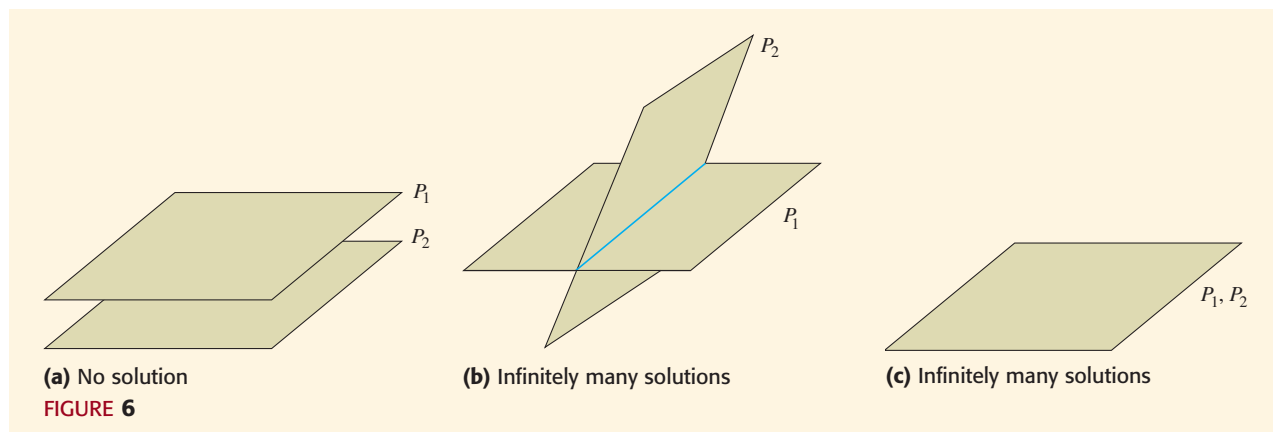
Thus, either there is no solution or there are infinitely many solutions corresponding to the points lying on a line of intersection of the two planes or on a single plane determined by the two equations. In the case where two planes intersect in a straight line, the solutions will involve one parameter, and in the case where the two planes are coincident, the solutions will involve two parameters.

Explore & Discuss

Give a geometric interpretation of Theorem 1 for a linear system composed of equations involving two variables. Specifically, illustrate what can happen if there are three linear equations in the system (the case involving two linear equations has already been discussed in Section 2.1). What if there are four linear equations? What if there is only one linear equation in the system?

EXAMPLE 3 A System with More Equations Than Variables Solve the following system of linear equations:

$$
\begin{aligned}
x + 2y &= 4 \\
x - 2y &= 0 \\
4x + 3y &= 12
\end{aligned}
$$

Solution   We obtain the following sequence of equivalent augmented matrices:

$$
\left[\begin{array}{cc|c} ① & 2 & 4 \\ 1 & -2 & 0 \\ 4 & 3 & 12 \end{array}\right]
\xrightarrow[R_3 - 4R_1]{R_2 - R_1}
\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & ⓸ & -4 \\ 0 & -5 & -4 \end{array}\right]
\xrightarrow{-\frac{1}{4}R_2}
$$

$$
\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -5 & -4 \end{array}\right]
\xrightarrow[R_3 + 5R_2]{R_1 - 2R_2}
\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]
$$

The last row of the row-reduced augmented matrix implies that $0 = 1$, which is impossible, so we conclude that the given system has no solution. Geometrically, the three lines defined by the three equations in the system do not intersect at a point. (To see this for yourself, draw the graphs of these equations.)   ◼

EXAMPLE 4 A System with More Variables Than Equations Solve the following system of linear equations:

$$
\begin{aligned}
x + 2y - 3z + w &= -2 \\
3x - y - 2z - 4w &= 1 \\
2x + 3y - 5z + w &= -3
\end{aligned}
$$

Solution   First, observe that the given system consists of three equations in four variables and so, by Theorem 1b, either the system has no solution or it has infinitely many solutions. To solve it we use the Gauss–Jordan method and obtain the following sequence of equivalent augmented matrices:

$$
\left[\begin{array}{cccc|c} ① & 2 & -3 & 1 & -2 \\ 3 & -1 & -2 & -4 & 1 \\ 2 & 3 & -5 & 1 & -3 \end{array}\right]
\xrightarrow[R_3 - 2R_1]{R_2 - 3R_1}
\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & -2 \\ 0 & ⓸{-7} & 7 & -7 & 7 \\ 0 & -1 & 1 & -1 & 1 \end{array}\right]
\xrightarrow{-\frac{1}{7}R_2}
$$

$$
\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & -2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 & 1 \end{array}\right]
\xrightarrow[R_3 + R_2]{R_1 - 2R_2}
\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

The last augmented matrix is in row-reduced form. Observe that the given system is equivalent to the system

$$\begin{aligned} x - z - w &= 0 \\ y - z + w &= -1 \end{aligned}$$

of two equations in four variables. Thus, we may solve for two of the variables in terms of the other two. Letting $z = s$ and $w = t$ (where s and t are any real numbers), we find that

$$\begin{aligned} x &= s + t \\ y &= s - t - 1 \\ z &= s \\ w &= t \end{aligned}$$

The solutions may be written in the form $(s + t, s - t - 1, s, t)$. Geometrically, the three equations in the system represent three hyperplanes in four-dimensional space (since there are four variables) and their "points" of intersection lie in a two-dimensional subspace of four-space (since there are two parameters). ■

Note In Example 4, we assigned parameters to z and w rather than to x and y because x and y are readily solved in terms of z and w. ■

The following example illustrates a situation in which a system of linear equations has infinitely many solutions.

APPLIED EXAMPLE 5 Traffic Control Figure 7 shows the flow of downtown traffic in a certain city during the rush hours on a typical weekday. The arrows indicate the direction of traffic flow on each one-way road, and the average number of vehicles per hour entering and leaving each intersection appears beside each road. 5th Avenue and 6th Avenue can each handle up to 2000 vehicles per hour without causing congestion, whereas the maximum capacity of both 4th Street and 5th Street is 1000 vehicles per hour. The flow of traffic is controlled by traffic lights installed at each of the four intersections.
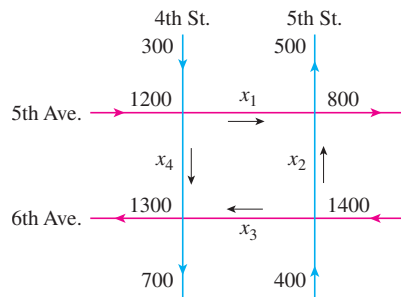
FIGURE **7**

a. Write a general expression involving the rates of flow—x_1, x_2, x_3, x_4—and suggest two possible flow patterns that will ensure no traffic congestion.
b. Suppose the part of 4th Street between 5th Avenue and 6th Avenue is to be resurfaced and that traffic flow between the two junctions must therefore be reduced to at most 300 vehicles per hour. Find two possible flow patterns that will result in a smooth flow of traffic.

Solution

a. To avoid congestion, all traffic entering an intersection must also leave that intersection. Applying this condition to each of the four intersections in a

clockwise direction beginning with the 5th Avenue and 4th Street intersection, we obtain the following equations:

$$1500 = x_1 + x_4$$
$$1300 = x_1 + x_2$$
$$1800 = x_2 + x_3$$
$$2000 = x_3 + x_4$$

This system of four linear equations in the four variables x_1, x_2, x_3, x_4 may be rewritten in the more standard form

$$
\begin{aligned}
x_1 \qquad\qquad + x_4 &= 1500 \\
x_1 + x_2 \qquad\qquad &= 1300 \\
x_2 + x_3 \qquad &= 1800 \\
x_3 + x_4 &= 2000
\end{aligned}
$$

Using the Gauss–Jordan elimination method to solve the system, we obtain

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & | & 1500 \\
1 & 1 & 0 & 0 & | & 1300 \\
0 & 1 & 1 & 0 & | & 1800 \\
0 & 0 & 1 & 1 & | & 2000
\end{bmatrix}
\xrightarrow{R_2 - R_1}
\begin{bmatrix}
1 & 0 & 0 & 1 & | & 1500 \\
0 & 1 & 0 & -1 & | & -200 \\
0 & 1 & 1 & 0 & | & 1800 \\
0 & 0 & 1 & 1 & | & 2000
\end{bmatrix}
$$

$$
\xrightarrow{R_3 - R_2}
\begin{bmatrix}
1 & 0 & 0 & 1 & | & 1500 \\
0 & 1 & 0 & -1 & | & -200 \\
0 & 0 & 1 & 1 & | & 2000 \\
0 & 0 & 1 & 1 & | & 2000
\end{bmatrix}
$$

$$
\xrightarrow{R_4 - R_3}
\begin{bmatrix}
1 & 0 & 0 & 1 & | & 1500 \\
0 & 1 & 0 & -1 & | & -200 \\
0 & 0 & 1 & 1 & | & 2000 \\
0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
$$

The last augmented matrix is in row-reduced form and is equivalent to a system of three linear equations in the four variables x_1, x_2, x_3, x_4. Thus, we may express three of the variables—say, x_1, x_2, x_3—in terms of x_4. Setting $x_4 = t$ (*t* a parameter), we may write the infinitely many solutions of the system as

$$
\begin{aligned}
x_1 &= 1500 - t \\
x_2 &= -200 + t \\
x_3 &= 2000 - t \\
x_4 &= t
\end{aligned}
$$

Observe that for a meaningful solution we must have $200 \le t \le 1000$ since $x_1, x_2, x_3,$ and x_4 must all be nonnegative and the maximum capacity of a street is 1000. For example, picking $t = 300$ gives the flow pattern

$$x_1 = 1200 \qquad x_2 = 100 \qquad x_3 = 1700 \qquad x_4 = 300$$

Selecting $t = 500$ gives the flow pattern

$$x_1 = 1000 \qquad x_2 = 300 \qquad x_3 = 1500 \qquad x_4 = 500$$

b. In this case, x_4 must not exceed 300. Again, using the results of part (a), we find, upon setting $x_4 = t = 300$, the flow pattern

$$x_1 = 1200 \qquad x_2 = 100 \qquad x_3 = 1700 \qquad x_4 = 300$$

obtained earlier. Picking $t = 250$ gives the flow pattern

$$x_1 = 1250 \qquad x_2 = 50 \qquad x_3 = 1750 \qquad x_4 = 250$$

2.3 Self-Check Exercises

1. The following augmented matrix in row-reduced form is equivalent to the augmented matrix of a certain system of linear equations. Use this result to solve the system of equations.

$$\begin{bmatrix} 1 & 0 & -1 & | & 3 \\ 0 & 1 & 5 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

2. Solve the system of linear equations

$$\begin{aligned} 2x - 3y + z &= 6 \\ x + 2y + 4z &= -4 \\ x - 5y - 3z &= 10 \end{aligned}$$

using the Gauss–Jordan elimination method.

3. Solve the system of linear equations

$$\begin{aligned} x - 2y + 3z &= 9 \\ 2x + 3y - z &= 4 \\ x + 5y - 4z &= 2 \end{aligned}$$

using the Gauss–Jordan elimination method.

Solutions to Self-Check Exercises 2.3 can be found on page 99.

2.3 Concept Questions

1. a. If a system of linear equations has the same number of equations or more equations than variables, what can you say about the nature of its solution(s)?

  b. If a system of linear equations has fewer equations than variables, what can you say about the nature of its solution(s)?

2. A system consists of three linear equations in four variables. Can the system have a unique solution?

2.3 Exercises

In Exercises 1–12, given that the augmented matrix in row-reduced form is equivalent to the augmented matrix of a system of linear equations, (a) determine whether the system has a solution and (b) find the solution or solutions to the system, if they exist.

1. $\begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 2 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 4 \\ 0 & 0 & | & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 1 & | & 4 \\ 0 & 1 & 0 & | & -2 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 0 & 0 & 0 & | & 3 \\ 0 & 1 & 1 & 0 & | & -1 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & 0 & | & 1 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & 0 & | & -1 \\ 0 & 0 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

10. $\begin{bmatrix} 0 & 1 & 0 & 1 & | & 3 \\ 0 & 0 & 1 & -2 & | & 4 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 3 & 0 & | & 2 \\ 0 & 1 & -1 & 0 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & 3 & -1 & | & 4 \\ 0 & 1 & -2 & 3 & | & 2 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$

In Exercises 13–32, solve the system of linear equations, using the Gauss–Jordan elimination method.

13. $\begin{aligned} 2x - y &= 3 \\ x + 2y &= 4 \\ 2x + 3y &= 7 \end{aligned}$

14. $\begin{aligned} x + 2y &= 3 \\ 2x - 3y &= -8 \\ x - 4y &= -9 \end{aligned}$

15. $\begin{aligned} 3x - 2y &= -3 \\ 2x + y &= 3 \\ x - 2y &= -5 \end{aligned}$

16. $\begin{aligned} 2x + 3y &= 2 \\ x + 3y &= -2 \\ x - y &= 3 \end{aligned}$

17. $\begin{aligned} 3x - 2y &= 5 \\ -x + 3y &= -4 \\ 2x - 4y &= 6 \end{aligned}$

18. $\begin{aligned} 4x + 6y &= 8 \\ 3x - 2y &= -7 \\ x + 3y &= 5 \end{aligned}$

19. $x - 2y = 2$
$7x - 14y = 14$
$3x - 6y = 6$

20. $x + 2y + z = -2$
$-2x - 3y - z = 1$
$2x + 4y + 2z = -4$

21. $3x + 2y = 4$
$-\frac{3}{2}x - y = -2$
$6x + 4y = 8$

22. $3y + 2z = 4$
$2x - y - 3z = 3$
$2x + 2y - z = 7$

23. $2x_1 - x_2 + x_3 = -4$
$3x_1 - \frac{3}{2}x_2 + \frac{3}{2}x_3 = -6$
$-6x_1 + 3x_2 - 3x_3 = 12$

24. $x + y - 2z = -3$
$2x - y + 3z = 7$
$x - 2y + 5z = 0$

25. $x - 2y + 3z = 4$
$2x + 3y - z = 2$
$x + 2y - 3z = -6$

26. $x_1 - 2x_2 + x_3 = -3$
$2x_1 + x_2 - 2x_3 = 2$
$x_1 + 3x_2 - 3x_3 = 5$

27. $4x + y - z = 4$
$8x + 2y - 2z = 8$

28. $x_1 + 2x_2 + 4x_3 = 2$
$x_1 + x_2 + 2x_3 = 1$

29. $2x + y - 3z = 1$
$x - y + 2z = 1$
$5x - 2y + 3z = 6$

30. $3x - 9y + 6z = -12$
$x - 3y + 2z = -4$
$2x - 6y + 4z = 8$

31. $x + 2y - z = -4$
$2x + y + z = 7$
$x + 3y + 2z = 7$
$x - 3y + z = 9$

32. $3x - 2y + z = 4$
$x + 3y - 4z = -3$
$2x - 3y + 5z = 7$
$x - 8y + 9z = 10$

33. MANAGEMENT DECISIONS The management of Hartman Rent-A-Car has allocated $1,008,000 to purchase 60 new automobiles to add to their existing fleet of rental cars. The company will choose from compact, mid-sized, and full-sized cars costing $12,000, $19,200, and $26,400 each, respectively. Find formulas giving the options available to the company. Give two specific options. (*Note*: Your answers will *not* be unique.)

34. NUTRITION A dietitian wishes to plan a meal around three foods. The meal is to include 8800 units of vitamin A, 3380 units of vitamin C, and 1020 units of calcium. The number of units of the vitamins and calcium in each ounce of the foods is summarized in the following table:

	Food I	Food II	Food III
Vitamin A	400	1200	800
Vitamin C	110	570	340
Calcium	90	30	60

Determine the amount of each food the dietitian should include in the meal in order to meet the vitamin and calcium requirements.

35. NUTRITION Refer to Exercise 34. In planning for another meal, the dietitian changes the requirement of vitamin C from 3380 units to 2160 units. All other requirements remain the same. Show that such a meal cannot be planned around the same foods.

36. MANUFACTURING PRODUCTION SCHEDULE Ace Novelty manufactures "Giant Pandas," "Saint Bernards," and "Big Birds." Each Panda requires 1.5 yd^2 of plush, 30 ft^3 of stuffing, and 5 pieces of trim; each Saint Bernard requires 2 yd^2 of plush, 35 ft^3 of stuffing, and 8 pieces of trim; and each Big Bird requires 2.5 yd^2 of plush, 25 ft^3 of stuffing, and 15 pieces of trim. If 4700 yd^2 of plush, 65,000 ft^3 of stuffing, and 23,400 pieces of trim are available, how many of each of the stuffed animals should the company manufacture if all the material is to be used? Give two specific options.

37. INVESTMENTS Mr. and Mrs. Garcia have a total of $100,000 to be invested in stocks, bonds, and a money market account. The stocks have a rate of return of 12%/year, while the bonds and the money market account pay 8%/year and 4%/year, respectively. The Garcias have stipulated that the amount invested in stocks should be equal to the sum of the amount invested in bonds and 3 times the amount invested in the money market account. How should the Garcias allocate their resources if they require an annual income of $10,000 from their investments? Give two specific options.

38. TRAFFIC CONTROL The accompanying figure shows the flow of traffic near a city's Civic Center during the rush hours on a typical weekday. Each road can handle a maximum of 1000 cars/hour without causing congestion. The flow of traffic is controlled by traffic lights at each of the five intersections.
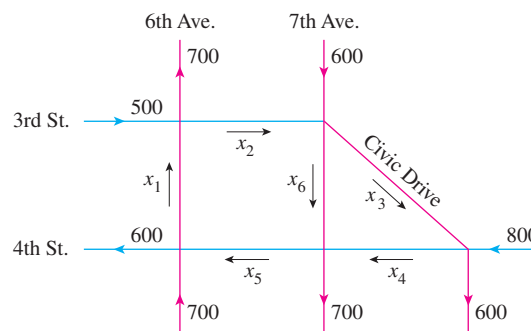
a. Set up a system of linear equations describing the traffic flow.
b. Solve the system devised in part (a) and suggest two possible traffic-flow patterns that will ensure no traffic congestion.
c. Suppose 7th Avenue between 3rd and 4th Streets is soon to be closed for road repairs. Find one possible flow pattern that will result in a smooth flow of traffic.

39. TRAFFIC CONTROL The accompanying figure shows the flow of downtown traffic during the rush hours on a typical weekday. Each avenue can handle up to 1500 vehicles/hour without causing congestion, whereas the maximum capacity of each street is 1000 vehicles/hour. The flow of traffic is controlled by traffic lights at each of the six intersections.

5th Ave.   6th Ave.

a. Set up a system of linear equations describing the traffic flow.
b. Solve the system devised in part (a) and suggest two possible traffic-flow patterns that will ensure no traffic congestion.
c. Suppose the traffic flow along 9th Street between 5th and 6th Avenues, x_6, is restricted because of sewer construction. What is the minimum permissible traffic flow along this road that will not result in traffic congestion?

40. Determine the value of k such that the following system of linear equations has a solution, and then find the solution:
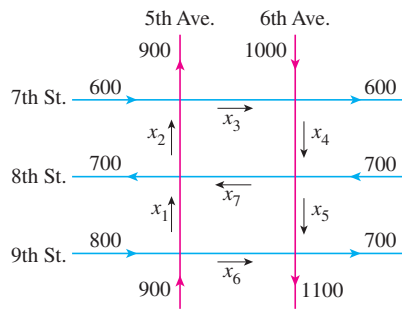
$$2x + 3y = 2$$
$$x + 4y = 6$$
$$5x + ky = 2$$

41. Determine the value of k such that the following system of linear equations has infinitely many solutions, and then find the solutions:

$$3x - 2y + 4z = 12$$
$$-9x + 6y - 12z = k$$

42. Solve the system:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -1$$
$$\frac{2}{x} + \frac{3}{y} + \frac{2}{z} = 3$$
$$\frac{2}{x} + \frac{1}{y} + \frac{2}{z} = -7$$

In Exercises 43 and 44, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

43. A system of linear equations having fewer equations than variables has no solution, a unique solution, or infinitely many solutions.

44. A system of linear equations having more equations than variables has no solution, a unique solution, or infinitely many solutions.

2.3 Solutions to Self-Check Exercises

1. Let x, y, and z denote the variables. Then, the given row-reduced augmented matrix tells us that the system of linear equations is equivalent to the two equations

$$x \quad - z = 3$$
$$y + 5z = -2$$

Letting $z = t$, where t is a parameter, we find the infinitely many solutions given by

$$x = t + 3$$
$$y = -5t - 2$$
$$z = t$$

2. We obtain the following sequence of equivalent augmented matrices:

$$\begin{bmatrix} 2 & -3 & 1 & | & 6 \\ 1 & 2 & 4 & | & -4 \\ 1 & -5 & -3 & | & 10 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\begin{bmatrix} 1 & 2 & 4 & | & -4 \\ 2 & -3 & 1 & | & 6 \\ 1 & -5 & -3 & | & 10 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & 2 & 4 & | & -4 \\ 0 & -7 & -7 & | & 14 \\ 0 & -7 & -7 & | & 14 \end{bmatrix} \xrightarrow{-\frac{1}{7}R_2}$$

$$\begin{bmatrix} 1 & 2 & 4 & | & -4 \\ 0 & 1 & 1 & | & -2 \\ 0 & -7 & -7 & | & 14 \end{bmatrix} \xrightarrow[R_3 + 7R_2]{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & 1 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The last augmented matrix, which is in row-reduced form, tells us that the given system of linear equations is equivalent to the following system of two equations:

$$x \quad + 2z = 0$$
$$y + z = -2$$

Letting $z = t$, where t is a parameter, we see that the infinitely many solutions are given by

$$x = -2t$$
$$y = -t - 2$$
$$z = t$$

3. We obtain the following sequence of equivalent augmented matrices:

$$\begin{bmatrix} \textcircled{1} & -2 & 3 & | & 9 \\ 2 & 3 & -1 & | & 4 \\ 1 & 5 & -4 & | & 2 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 - 2R_1}$$

$$\begin{bmatrix} 1 & -2 & 3 & | & 9 \\ 0 & 7 & -7 & | & -14 \\ 0 & 7 & -7 & | & -7 \end{bmatrix} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & -2 & 3 & | & 9 \\ 0 & 7 & -7 & | & -14 \\ 0 & 0 & 0 & | & 7 \end{bmatrix}$$

Since the last row of the final augmented matrix is equivalent to the equation $0 = 7$, a contradiction, we conclude that the given system has no solution.

Systems of Linear Equations: Underdetermined and Overdetermined Systems

We can use the row operations of a graphing utility to solve a system of m linear equations in n unknowns by the Gauss–Jordan method, as we did in the previous technology section. We can also use the **rref** or equivalent operation to obtain the row-reduced form without going through all the steps of the Gauss–Jordan method. The SIMULT function, however, cannot be used to solve a system where the number of equations and the number of variables are not the same.

EXAMPLE 1 Solve the system

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= -1 \\ 3x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + 6x_2 - 12x_3 &= -6 \end{aligned}$$

Solution First, we enter the augmented matrix A into the calculator as

$$A = \begin{bmatrix} 1 & -2 & 4 & | & 2 \\ 2 & 1 & -2 & | & -1 \\ 3 & -1 & 2 & | & 1 \\ 2 & 6 & -12 & | & -6 \end{bmatrix}$$

Then using the **rref** or equivalent operation, we obtain the equivalent matrix

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -2 & | & -1 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

in reduced form. Thus, the given system is equivalent to

$$\begin{aligned} x_1 \qquad\qquad &= 0 \\ x_2 - 2x_3 &= -1 \end{aligned}$$

If we let $x_3 = t$, where t is a parameter, then we find that the solutions are $(0, 2t - 1, t)$. ∎

TECHNOLOGY EXERCISES

Use a graphing utility to solve the system of equations using the rref or equivalent operation.

1. $\begin{aligned} 2x_1 - x_2 - x_3 &= 0 \\ 3x_1 - 2x_2 - x_3 &= -1 \\ -x_1 + 2x_2 - x_3 &= 3 \\ 2x_2 - 2x_3 &= 4 \end{aligned}$

2. $\begin{aligned} 3x_1 + x_2 - 4x_3 &= 5 \\ 2x_1 - 3x_2 + 2x_3 &= -4 \\ -x_1 - 2x_2 + 4x_3 &= 6 \\ 4x_1 + 3x_2 - 5x_3 &= 9 \end{aligned}$

3. $\begin{aligned} 2x_1 + 3x_2 + 2x_3 + x_4 &= -1 \\ x_1 - x_2 + x_3 - 2x_4 &= -8 \\ 5x_1 + 6x_2 - 2x_3 + 2x_4 &= 11 \\ x_1 + 3x_2 + 8x_3 + x_4 &= -14 \end{aligned}$

4. $\begin{aligned} x_1 - x_2 + 3x_3 - 6x_4 &= 2 \\ x_1 + x_2 + x_3 - 2x_4 &= 2 \\ -2x_1 - x_2 + x_3 + 2x_4 &= 0 \end{aligned}$

5. $\begin{aligned} x_1 + x_2 - x_3 - x_4 &= -1 \\ x_1 - x_2 + x_3 + 4x_4 &= -6 \\ 3x_1 + x_2 - x_3 + 2x_4 &= -4 \\ 5x_1 + x_2 - 3x_3 + x_4 &= -9 \end{aligned}$

6. $\begin{aligned} 1.2x_1 - 2.3x_2 + 4.2x_3 + 5.4x_4 - 1.6x_5 &= 4.2 \\ 2.3x_1 + 1.4x_2 - 3.1x_3 + 3.3x_4 - 2.4x_5 &= 6.3 \\ 1.7x_1 + 2.6x_2 - 4.3x_3 + 7.2x_4 - 1.8x_5 &= 7.8 \\ 2.6x_1 - 4.2x_2 + 8.3x_3 - 1.6x_4 + 2.5x_5 &= 6.4 \end{aligned}$

2.4 Matrices

Using Matrices to Represent Data

Many practical problems are solved by using arithmetic operations on the data associated with the problems. By properly organizing the data into *blocks* of numbers, we can then carry out these arithmetic operations in an orderly and efficient manner. In particular, this systematic approach enables us to use the computer to full advantage.

Let's begin by considering how the monthly output data of a manufacturer may be organized. The Acrosonic Company manufactures four different loudspeaker systems at three separate locations. The company's May output is described in Table 1.

TABLE 1

	Model A	Model B	Model C	Model D
Location I	320	280	460	280
Location II	480	360	580	0
Location III	540	420	200	880

Now, if we agree to preserve the relative location of each entry in Table 1, we can summarize the set of data as follows:

$$\begin{bmatrix} 320 & 280 & 460 & 280 \\ 480 & 360 & 580 & 0 \\ 540 & 420 & 200 & 880 \end{bmatrix}$$

A matrix summarizing the data in Table 1

The array of numbers displayed here is an example of a matrix. Observe that the numbers in row 1 give the output of models A, B, C, and D of Acrosonic loudspeaker systems manufactured at location I; similarly, the numbers in rows 2 and 3 give the respective outputs of these loudspeaker systems at locations II and III. The numbers in each column of the matrix give the outputs of a particular model of loudspeaker system manufactured at each of the company's three manufacturing locations.

More generally, a matrix is a rectangular array of real numbers. For example, each of the following arrays is a matrix:

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ -1 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 3 & 0 & 1 \end{bmatrix}$$

The real numbers that make up the array are called the **entries,** or *elements*, of the matrix. The entries in a row in the array are referred to as a **row** of the matrix, whereas the entries in a column in the array are referred to as a **column** of the matrix. Matrix A, for example, has two rows and three columns, which may be identified as follows:

$$\begin{array}{cccc} & \text{Column 1} & \text{Column 2} & \text{Column 3} \\ \text{Row 1} & \begin{bmatrix} 3 & 0 & -1 \\ \text{Row 2} & 2 & 1 & 4 \end{bmatrix} \end{array}$$

A 2×3 matrix

The **size,** or *dimension*, **of a matrix** is described in terms of the number of rows and columns of the matrix. For example, matrix A has two rows and three columns and is said to have size 2 by 3, denoted 2×3. In general, a matrix having m rows and n columns is said to have size $m \times n$.

> ### Matrix
>
> A **matrix** is an ordered rectangular array of numbers. A matrix with m rows and n columns has size $m \times n$. The entry in the ith row and jth column of a matrix A is denoted by a_{ij}.

A matrix of size $1 \times n$—a matrix having one row and n columns—is referred to as a **row matrix,** or *row vector*, of dimension n. For example, the matrix D is a row vector of dimension 4. Similarly, a matrix having m rows and one column is referred to as a **column matrix,** or *column vector*, of dimension m. The matrix C is a column vector of dimension 4. Finally, an $n \times n$ matrix—that is, a matrix having the same number of rows as columns—is called a **square matrix.** For example, the matrix

$$\begin{bmatrix} -3 & 8 & 6 \\ 2 & \frac{1}{4} & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

A 3×3 square matrix

is a square matrix of size 3×3, or simply of size 3.

APPLIED EXAMPLE 1 **Organizing Production Data** Consider the matrix

$$P = \begin{bmatrix} 320 & 280 & 460 & 280 \\ 480 & 360 & 580 & 0 \\ 540 & 420 & 200 & 880 \end{bmatrix}$$

representing the output of loudspeaker systems of the Acrosonic Company discussed earlier (see Table 1).

a. What is the size of the matrix P?
b. Find p_{24} (the entry in row 2 and column 4 of the matrix P) and give an interpretation of this number.

   c. Find the sum of the entries that make up row 1 of P and interpret the result.

   d. Find the sum of the entries that make up column 4 of P and interpret the result.

Solution

   a. The matrix P has three rows and four columns and hence has size 3×4.

   b. The required entry lies in row 2 and column 4, and is the number 0. This means that no model D loudspeaker system was manufactured at location II in May.

   c. The required sum is given by

$$320 + 280 + 460 + 280 = 1340$$

which gives the total number of loudspeaker systems manufactured at location I in May as 1340 units.

   d. The required sum is given by

$$280 + 0 + 880 = 1160$$

giving the output of model D loudspeaker systems at all locations of the company in May as 1160 units.   ■

Equality of Matrices

Two matrices are said to be *equal* if they have the same size and their corresponding entries are equal. For example,

$$\begin{bmatrix} 2 & 3 & 1 \\ 4 & 6 & 2 \end{bmatrix} = \begin{bmatrix} (3-1) & 3 & 1 \\ 4 & (4+2) & 2 \end{bmatrix}$$

Also,

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 3 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 3 \end{bmatrix}$$

since the matrix on the left has size 2×3 whereas the matrix on the right has size 3×2, and

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

since the corresponding elements in row 2 and column 2 of the two matrices are not equal.

> ### Equality of Matrices
>
> Two matrices are equal if they have the same size and their corresponding entries are equal.

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EXAMPLE 2 Solve the following matrix equation for x, y, and z:

$$\begin{bmatrix} 1 & x & 3 \\ 2 & y-1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & z \\ 2 & 1 & 2 \end{bmatrix}$$

Solution   Since the corresponding elements of the two matrices must be equal, we find that $x = 4$, $z = 3$, and $y - 1 = 1$, or $y = 2$.   ■

Addition and Subtraction

Two matrices A and B of the *same size* can be added or subtracted to produce a matrix of the same size. This is done by adding or subtracting the corresponding entries in the two matrices. For example,

$$\begin{bmatrix} 1 & 3 & 4 \\ -1 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 3 \\ 6 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 1+1 & 3+4 & 4+3 \\ -1+6 & 2+1 & 0+(-2) \end{bmatrix} = \begin{bmatrix} 2 & 7 & 7 \\ 5 & 3 & -2 \end{bmatrix}$$

Adding two matrices of the same size

$$\text{and} \quad \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1-2 & 2-(-1) \\ -1-3 & 3-2 \\ 4-(-1) & 0-0 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ -4 & 1 \\ 5 & 0 \end{bmatrix}$$

Subtracting two matrices of the same size

Addition and Subtraction of Matrices

If A and B are two matrices of the same size, then:

1. The *sum $A + B$* is the matrix obtained by adding the corresponding entries in the two matrices.

2. The *difference $A - B$* is the matrix obtained by subtracting the corresponding entries in B from those in A.

APPLIED EXAMPLE 3 Organizing Production Data The total output of Acrosonic for June is shown in Table 2.

TABLE **2**

	Model A	Model B	Model C	Model D
Location I	210	180	330	180
Location II	400	300	450	40
Location III	420	280	180	740

The output for May was given earlier in Table 1. Find the total output of the company for May and June.

Solution   As we saw earlier, the production matrix for Acrosonic in May is given by

$$A = \begin{bmatrix} 320 & 280 & 460 & 280 \\ 480 & 360 & 580 & 0 \\ 540 & 420 & 200 & 880 \end{bmatrix}$$

Next, from Table 2, we see that the production matrix for June is given by

$$B = \begin{bmatrix} 210 & 180 & 330 & 180 \\ 400 & 300 & 450 & 40 \\ 420 & 280 & 180 & 740 \end{bmatrix}$$

Finally, the total output of Acrosonic for May and June is given by the matrix

$$A + B = \begin{bmatrix} 320 & 280 & 460 & 280 \\ 480 & 360 & 580 & 0 \\ 540 & 420 & 200 & 880 \end{bmatrix} + \begin{bmatrix} 210 & 180 & 330 & 180 \\ 400 & 300 & 450 & 40 \\ 420 & 280 & 180 & 740 \end{bmatrix}$$

$$= \begin{bmatrix} 530 & 460 & 790 & 460 \\ 880 & 660 & 1030 & 40 \\ 960 & 700 & 380 & 1620 \end{bmatrix}$$ ∎

The following laws hold for matrix addition.

> **Laws for Matrix Addition**
> If A, B, and C are matrices of the same size, then
> **1.** $A + B = B + A$ Commutative law
> **2.** $(A + B) + C = A + (B + C)$ Associative law

The *commutative law* for matrix addition states that the order in which matrix addition is performed is immaterial. The *associative law* states that, when adding three matrices together, we may first add A and B and then add the resulting sum to C. Equivalently, we can add A to the sum of B and C.

A *zero matrix* is one in which all entries are zero. A zero matrix O has the property that

$$A + O = O + A = A$$

for any matrix A having the same size as that of O. For example, the zero matrix of size 3×2 is

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If A is any 3×2 matrix, then

$$A + O = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = A$$

where a_{ij} denotes the entry in the ith row and jth column of the matrix A.

The matrix obtained by interchanging the rows and columns of a given matrix A is called the *transpose* of A and is denoted A^T. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

then

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

> **Transpose of a Matrix**
>
> If A is an $m \times n$ matrix with elements a_{ij}, then the **transpose** of A is the $n \times m$ matrix A^T with elements a_{ji}.

Scalar Multiplication

A matrix A may be multiplied by a real number, called a **scalar** in the context of matrix algebra. The scalar product, denoted by cA, is a matrix obtained by multiplying each entry of A by c. For example, the scalar product of the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

and the scalar 3 is the matrix

$$3A = 3\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 6 \\ 0 & 3 & 12 \end{bmatrix}$$

> **Scalar Product**
>
> If A is a matrix and c is a real number, then the **scalar product** cA is the matrix obtained by multiplying each entry of A by c.

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EXAMPLE 4 Given

$$A = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$$

find the matrix X satisfying the *matrix equation* $2X + B = 3A$.

Solution   From the given equation $2X + B = 3A$, we find that

$$2X = 3A - B$$
$$= 3\begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 12 \\ -3 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ -2 & 4 \end{bmatrix}$$
$$X = \frac{1}{2}\begin{bmatrix} 6 & 10 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$$

∎

APPLIED EXAMPLE 5 Production Planning The management of Acrosonic has decided to increase its July production of loudspeaker systems by 10% (over its June output). Find a matrix giving the targeted production for July.

Solution   From the results of Example 3, we see that Acrosonic's total output for June may be represented by the matrix

$$B = \begin{bmatrix} 210 & 180 & 330 & 180 \\ 400 & 300 & 450 & 40 \\ 420 & 280 & 180 & 740 \end{bmatrix}$$

The required matrix is given by

$$(1.1)B = 1.1\begin{bmatrix} 210 & 180 & 330 & 180 \\ 400 & 300 & 450 & 40 \\ 420 & 280 & 180 & 740 \end{bmatrix}$$

$$= \begin{bmatrix} 231 & 198 & 363 & 198 \\ 440 & 330 & 495 & 44 \\ 462 & 308 & 198 & 814 \end{bmatrix}$$

and is interpreted in the usual manner. ∎

2.4 Self-Check Exercises

1. Perform the indicated operations:

$$\begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 7 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix}$$

2. Solve the following matrix equation for x, y, and z:

$$\begin{bmatrix} x & 3 \\ z & 2 \end{bmatrix} + \begin{bmatrix} 2 - y & z \\ 2 - z & -x \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 2 & 0 \end{bmatrix}$$

3. Jack owns two gas stations, one downtown and the other in the Wilshire district. Over 2 consecutive days his gas stations recorded gasoline sales represented by the following matrices:

$$A = \begin{matrix} \text{Downtown} \\ \text{Wilshire} \end{matrix} \begin{matrix} \text{Regular} & \text{Regular plus} & \text{Premium} \\ \begin{bmatrix} 1200 & 750 & 650 \\ 1100 & 850 & 600 \end{bmatrix} \end{matrix}$$

and

$$B = \begin{matrix} \text{Downtown} \\ \text{Wilshire} \end{matrix} \begin{matrix} \text{Regular} & \text{Regular plus} & \text{Premium} \\ \begin{bmatrix} 1250 & 825 & 550 \\ 1150 & 750 & 750 \end{bmatrix} \end{matrix}$$

Find a matrix representing the total sales of the two gas stations over the 2-day period.

Solutions to Self-Check Exercises 2.4 can be found on page 110.

2.4 Concept Questions

1. Define (a) a matrix, (b) the size of a matrix, (c) a row matrix, (d) a column matrix, and (e) a square matrix.

2. When are two matrices equal? Give an example of two matrices that are equal.

3. Construct a 3×3 matrix A having the property that $A = A^T$. What special characteristic does A have?

2.4 Exercises

In Exercises 1–6, refer to the following matrices:

$$A = \begin{bmatrix} 2 & -3 & 9 & -4 \\ -11 & 2 & 6 & 7 \\ 6 & 0 & 2 & 9 \\ 5 & 1 & 5 & -8 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \\ -1 & 0 & 8 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 3 & 4 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$$

1. What is the size of A? Of B? Of C? Of D?

2. Find a_{14}, a_{21}, a_{31}, and a_{43}. **3.** Find b_{13}, b_{31}, and b_{43}.

4. Identify the row matrix. What is its transpose?

5. Identify the column matrix. What is its transpose?

6. Identify the square matrix. What is its transpose?

In Exercises 7–12, refer to the following matrices:

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -2 \\ 4 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ 3 & 1 \\ -2 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & 3 \\ 4 & 6 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & -2 & 4 \\ 3 & 6 & 2 \\ -2 & 3 & 1 \end{bmatrix}$$

7. What is the size of A? Of B? Of C? Of D?

8. Explain why the matrix $A + C$ does *not* exist.

9. Compute $A + B$.   **10.** Compute $2A - 3B$.

11. Compute $C - D$.   **12.** Compute $4D - 2C$.

In Exercises 13–20, perform the indicated operations.

13. $\begin{bmatrix} 6 & 3 & 8 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -1 \\ 0 & -5 & -7 \end{bmatrix}$

14. $\begin{bmatrix} 2 & -3 & 4 & -1 \\ 3 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 3 & -2 & -4 \\ 6 & 2 & 0 & -3 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 4 & -5 \\ 3 & -8 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 & -2 \\ 3 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 8 & 9 \\ -11 & 2 & -5 \end{bmatrix}$

16. $3\begin{bmatrix} 1 & 1 & -3 \\ 3 & 2 & 3 \\ 7 & -1 & 6 \end{bmatrix} + 4\begin{bmatrix} -2 & -1 & 8 \\ 4 & 2 & 2 \\ 3 & 6 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 1.2 & 4.5 & -4.2 \\ 8.2 & 6.3 & -3.2 \end{bmatrix} - \begin{bmatrix} 3.1 & 1.5 & -3.6 \\ 2.2 & -3.3 & -4.4 \end{bmatrix}$

18. $\begin{bmatrix} 0.06 & 0.12 \\ 0.43 & 1.11 \\ 1.55 & -0.43 \end{bmatrix} - \begin{bmatrix} 0.77 & -0.75 \\ 0.22 & -0.65 \\ 1.09 & -0.57 \end{bmatrix}$

19. $\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -4 \\ 3 & 0 & -1 & 6 \\ -2 & 1 & -4 & 2 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 3 & 0 & -1 & 4 \\ -2 & 1 & -6 & 2 \\ 8 & 2 & 0 & -2 \end{bmatrix}$

$-\frac{1}{3}\begin{bmatrix} 3 & -9 & -1 & 0 \\ 6 & 2 & 0 & -6 \\ 0 & 1 & -3 & 1 \end{bmatrix}$

20. $0.5\begin{bmatrix} 1 & 3 & 5 \\ 5 & 2 & -1 \\ -2 & 0 & 1 \end{bmatrix} - 0.2\begin{bmatrix} 2 & 3 & 4 \\ -1 & 1 & -4 \\ 3 & 5 & -5 \end{bmatrix}$

$+ 0.6\begin{bmatrix} 3 & 4 & -1 \\ 4 & 5 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

In Exercises 21–24, solve for *u*, *x*, *y*, and *z* in the given matrix equation.

21. $\begin{bmatrix} 2x - 2 & 3 & 2 \\ 2 & 4 & y - 2 \\ 2z & -3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & u & 2 \\ 2 & 4 & 5 \\ 4 & -3 & 2 \end{bmatrix}$

22. $\begin{bmatrix} x & -2 \\ 3 & y \end{bmatrix} + \begin{bmatrix} -2 & z \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2u & 4 \end{bmatrix}$

23. $\begin{bmatrix} 1 & x \\ 2y & -3 \end{bmatrix} - 4\begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3z & 10 \\ 4 & -u \end{bmatrix}$

24. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ x & -1 \end{bmatrix} - 3\begin{bmatrix} y - 1 & 2 \\ 1 & 2 \\ 4 & 2z + 1 \end{bmatrix} = 2\begin{bmatrix} -4 & -u \\ 0 & -1 \\ 4 & 4 \end{bmatrix}$

In Exercises 25 and 26, let

$$A = \begin{bmatrix} 2 & -4 & 3 \\ 4 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{bmatrix}$$

25. Verify by direct computation the validity of the commutative law for matrix addition.

26. Verify by direct computation the validity of the associative law for matrix addition.

In Exercises 27–30, let

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ -4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 3 & 2 \end{bmatrix}$$

Verify each equation by direct computation.

27. $(3 + 5)A = 3A + 5A$ **28.** $2(4A) = (2 \cdot 4)A = 8A$

29. $4(A + B) = 4A + 4B$ **30.** $2(A - 3B) = 2A - 6B$

In Exercises 31–34, find the transpose of each matrix.

31. $\begin{bmatrix} 3 & 2 & -1 & 5 \end{bmatrix}$ **32.** $\begin{bmatrix} 4 & 2 & 0 & -1 \\ 3 & 4 & -1 & 5 \end{bmatrix}$

33. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ **34.** $\begin{bmatrix} 1 & 2 & 6 & 4 \\ 2 & 3 & 2 & 5 \\ 6 & 2 & 3 & 0 \\ 4 & 5 & 0 & 2 \end{bmatrix}$

35. CHOLESTEROL LEVELS Mr. Cross, Mr. Jones, and Mr. Smith each suffer from coronary heart disease. As part of their treatment, they were put on special low-cholesterol diets: Cross on diet I, Jones on diet II, and Smith on diet III. Progressive records of each patient's cholesterol level were kept. At the beginning of the first, second, third, and fourth months, the cholesterol levels of the three patients were:

Cross: 220, 215, 210, and 205
Jones: 220, 210, 200, and 195
Smith: 215, 205, 195, and 190

Represent this information in a 3 × 4 matrix.

36. INVESTMENT PORTFOLIOS The following table gives the number of shares of certain corporations held by Leslie and Tom in their respective IRA accounts at the beginning of the year:

	IBM	GE	Ford	Wal-Mart
Leslie	500	350	200	400
Tom	400	450	300	200

Over the year, they added more shares to their accounts, as shown in the following table:

	IBM	GE	Ford	Wal-Mart
Leslie	50	50	0	100
Tom	0	80	100	50

a. Write a matrix A giving the holdings of Leslie and Tom at the beginning of the year and a matrix B giving the shares they have added to their portfolios.
b. Find a matrix C giving their total holdings at the end of the year.

37. HOME SALES K & R Builders build three models of houses, M_1, M_2, and M_3, in three subdivisions I, II, and III located in three different areas of a city. The prices of the houses (in thousands of dollars) are given in matrix A:

$$A = \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} 340 & 360 & 380 \\ 410 & 430 & 440 \\ 620 & 660 & 700 \end{bmatrix}$$

K & R Builders has decided to raise the price of each house by 3% next year. Write a matrix B giving the new prices of the houses.

38. HOME SALES K & R Builders build three models of houses, M_1, M_2, and M_3, in three subdivisions I, II, and III located in three different areas of a city. The prices of the homes (in thousands of dollars) are given in matrix A:

$$A = \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} 340 & 360 & 380 \\ 410 & 430 & 440 \\ 620 & 660 & 700 \end{bmatrix}$$

The new price schedule for next year, reflecting a uniform percentage increase in each house, is given by matrix B:

$$B = \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{bmatrix} 357 & 378 & 399 \\ 430.5 & 451.5 & 462 \\ 651 & 693 & 735 \end{bmatrix}$$

What was the percentage increase in the prices of the houses?
Hint: Find r such that $(1 + 0.01r)A = B$.

39. BANKING The numbers of three types of bank accounts on January 1 at the Central Bank and its branches are represented by matrix A:

$$A = \begin{array}{c} \text{Main office} \\ \text{Westside branch} \\ \text{Eastside branch} \end{array} \begin{bmatrix} 2820 & 1470 & 1120 \\ 1030 & 520 & 480 \\ 1170 & 540 & 460 \end{bmatrix}$$

with columns: Checking accounts, Savings accounts, Fixed-deposit accounts.

The number and types of accounts opened during the first quarter are represented by matrix B, and the number and types of accounts closed during the same period are represented by matrix C. Thus,

$$B = \begin{bmatrix} 260 & 120 & 110 \\ 140 & 60 & 50 \\ 120 & 70 & 50 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 120 & 80 & 80 \\ 70 & 30 & 40 \\ 60 & 20 & 40 \end{bmatrix}$$

a. Find matrix D, which represents the number of each type of account at the end of the first quarter at each location.
b. Because a new manufacturing plant is opening in the immediate area, it is anticipated that there will be a 10% increase in the number of accounts at each location during the second quarter. Write a matrix $E = 1.1 D$ to reflect this anticipated increase.

40. BOOKSTORE INVENTORIES The Campus Bookstore's inventory of books is

Hardcover: textbooks, 5280; fiction, 1680; nonfiction, 2320; reference, 1890

Paperback: fiction, 2810; nonfiction, 1490; reference, 2070; textbooks, 1940

The College Bookstore's inventory of books is

Hardcover: textbooks, 6340; fiction, 2220; nonfiction, 1790; reference, 1980

Paperback: fiction, 3100; nonfiction, 1720; reference, 2710; textbooks, 2050

a. Represent Campus's inventory as a matrix A.
b. Represent College's inventory as a matrix B.
c. The two companies decide to merge, so now write a matrix C that represents the total inventory of the newly amalgamated company.

41. INSURANCE CLAIMS The property damage claim frequencies per 100 cars in Massachusetts in the years 2000, 2001, and 2002 are 6.88, 7.05, and 7.18, respectively. The corresponding claim frequencies in the United States are 4.13, 4.09, and 4.06, respectively. Express this information using a 2 × 3 matrix.

Sources: Registry of Motor Vehicles; Federal Highway Administration

42. MORTALITY RATES Mortality actuarial tables in the United States were revised in 2001, the fourth time since 1858. Based on the new life insurance mortality rates, 1% of 60-yr-old men, 2.6% of 70-yr-old men, 7% of 80-yr-old men, 18.8% of 90-yr-old men, and 36.3% of 100-yr-old men would die within a year. The corresponding rates for women are 0.8%, 1.8%, 4.4%, 12.2%, and 27.6%, respectively. Express this information using a 2×5 matrix.

Source: Society of Actuaries

43. LIFE EXPECTANCY Figures for life expectancy at birth of Massachusetts residents in 2002 are 81.0, 76.1, and 82.2 yr for white, black, and Hispanic women, respectively, and 76.0, 69.9, and 75.9 years for white, black, and Hispanic men, respectively. Express this information using a 2×3 matrix and a 3×2 matrix.

Source: Massachusetts Department of Public Health

44. MARKET SHARE OF MOTORCYCLES The market share of motorcycles in the United States in 2001 follows: Honda 27.9%, Harley-Davidson 21.9%, Yamaha 19.2%, Suzuki 11.0%, Kawasaki 9.1%, and others 10.9%. The corre- sponding figures for 2002 are 27.6%, 23.3%, 18.2%, 10.5%, 8.8%, and 11.6%, respectively. Express this infor- mation using a 2×6 matrix. What is the sum of all the ele- ments in the first row? In the second row? Is this expected? Which company gained the most market share between 2001 and 2002?

Source: Motorcycle Industry Council

In Exercises 45–48, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

45. If A and B are matrices of the same size and c is a scalar, then $c(A + B) = cA + cB$.

46. If A and B are matrices of the same size, then $A - B = A + (-1)B$.

47. If A is a matrix and c is a nonzero scalar, then $(cA)^T = (1/c)A^T$.

48. If A is a matrix, then $(A^T)^T = A$.

2.4 Solutions to Self-Check Exercises

1. $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 7 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 4 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 3 & 0 \\ 3 & 9 & 12 \end{bmatrix}$

$= \begin{bmatrix} -5 & 0 & 2 \\ -4 & -5 & -5 \end{bmatrix}$

2. We are given

$\begin{bmatrix} x & 3 \\ z & 2 \end{bmatrix} + \begin{bmatrix} 2 - y & z \\ 2 - z & -x \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 2 & 0 \end{bmatrix}$

Performing the indicated operation on the left-hand side, we obtain

$\begin{bmatrix} 2 + x - y & 3 + z \\ 2 & 2 - x \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 2 & 0 \end{bmatrix}$

By the equality of matrices, we have

$$2 + x - y = 3$$
$$3 + z = 7$$
$$2 - x = 0$$

from which we deduce that $x = 2$, $y = 1$, and $z = 4$.

3. The required matrix is

$A + B = \begin{bmatrix} 1200 & 750 & 650 \\ 1100 & 850 & 600 \end{bmatrix} + \begin{bmatrix} 1250 & 825 & 550 \\ 1150 & 750 & 750 \end{bmatrix}$

$= \begin{bmatrix} 2450 & 1575 & 1200 \\ 2250 & 1600 & 1350 \end{bmatrix}$

USING TECHNOLOGY

Matrix Operations

Graphing Utility

A graphing utility can be used to perform matrix addition, matrix subtraction, and scalar multiplication. It can also be used to find the transpose of a matrix.

EXAMPLE 1 Let

$$A = \begin{bmatrix} 1.2 & 3.1 \\ -2.1 & 4.2 \\ 3.1 & 4.8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4.1 & 3.2 \\ 1.3 & 6.4 \\ 1.7 & 0.8 \end{bmatrix}$$

Find (a) $A + B$, (b) $2.1A - 3.2B$, and (c) $(2.1A + 3.2B)^T$.

Solution We first enter the matrices A and B into the calculator.

a. Using matrix operations, we enter the expression $A + B$ and obtain

$$A + B = \begin{bmatrix} 5.3 & 6.3 \\ -0.8 & 10.6 \\ 4.8 & 5.6 \end{bmatrix}$$

b. Using matrix operations, we enter the expression $2.1A - 3.2B$ and obtain

$$2.1A - 3.2B = \begin{bmatrix} -10.6 & -3.73 \\ -8.57 & -11.66 \\ 1.07 & 7.52 \end{bmatrix}$$

c. Using matrix operations, we enter the expression $(2.1A + 3.2B)^T$ and obtain

$$(2.1A + 3.2B)^T = \begin{bmatrix} 15.64 & -0.25 & 11.95 \\ 16.75 & 29.3 & 12.64 \end{bmatrix}$$
◾

APPLIED EXAMPLE 2 John operates three gas stations at three locations, I, II, and III. Over 2 consecutive days, his gas stations recorded the following fuel sales (in gallons):

| | **Day 1** | | | |
	Regular	**Regular Plus**	**Premium**	**Diesel**
Location I	1400	1200	1100	200
Location II	1600	900	1200	300
Location III	1200	1500	800	500

| | **Day 2** | | | |
	Regular	**Regular Plus**	**Premium**	**Diesel**
Location I	1000	900	800	150
Location II	1800	1200	1100	250
Location III	800	1000	700	400

Find a matrix representing the total fuel sales at John's gas stations.

Solution The fuel sales can be represented by the matrix A (day 1) and matrix B (day 2):

$$A = \begin{bmatrix} 1400 & 1200 & 1100 & 200 \\ 1600 & 900 & 1200 & 300 \\ 1200 & 1500 & 800 & 500 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1000 & 900 & 800 & 150 \\ 1800 & 1200 & 1100 & 250 \\ 800 & 1000 & 700 & 400 \end{bmatrix}$$

We enter the matrices A and B into the calculator. Using matrix operations, we enter the expression $A + B$ and obtain

$$A + B = \begin{bmatrix} 2400 & 2100 & 1900 & 350 \\ 3400 & 2100 & 2300 & 550 \\ 2000 & 2500 & 1500 & 900 \end{bmatrix}$$
◾

Excel

First, we show how basic operations on matrices can be carried out using Excel.

EXAMPLE 3 Given the following matrices,

$$A = \begin{bmatrix} 1.2 & 3.1 \\ -2.1 & 4.2 \\ 3.1 & 4.8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4.1 & 3.2 \\ 1.3 & 6.4 \\ 1.7 & 0.8 \end{bmatrix}$$

a. Compute $A + B$. **b.** Compute $2.1A - 3.2B$.

Solution

a. *First, represent the matrices A and B in a spreadsheet.* Enter the elements of each matrix in a block of cells as shown in Figure T1.

	A	B	C	D	E
1		A			B
2	1.2	3.1		4.1	3.2
3	−2.1	4.2		1.3	6.4
4	3.1	4.8		1.7	0.8

FIGURE T1
The elements of matrix *A* and matrix *B* in a spreadsheet

Second, compute the sum of matrix A and matrix B. Highlight the cells that will contain matrix $A + B$, type =, highlight the cells in matrix *A*, type +, highlight the cells in matrix *B*, and press **Ctrl-Shift-Enter**. The resulting matrix $A + B$ is shown in Figure T2.

	A	B
8		A + B
9	5.3	6.3
10	−0.8	10.6
11	4.8	5.6

FIGURE T2
The matrix *A* + *B*

b. *Highlight the cells that will contain matrix* $2.1A - 3.2B$. Type = 2.1*, highlight matrix *A*, type −3.2*, highlight the cells in matrix *B*, and press **Ctrl-Shift-Enter**. The resulting matrix $2.1A - 3.2B$ is shown in Figure T3.

	A	B
13		2.1A - 3.2B
14	−10.6	−3.73
15	−8.57	−11.66
16	1.07	7.52

FIGURE T3
The matrix $2.1A - 3.2B$

APPLIED EXAMPLE 4 John operates three gas stations at three locations I, II, and III. Over 2 consecutive days, his gas stations recorded the following fuel sales (in gallons):

| | Day 1 | | | |
	Regular	Regular Plus	Premium	Diesel
Location I	1400	1200	1100	200
Location II	1600	900	1200	300
Location III	1200	1500	800	500

| | Day 2 | | | |
	Regular	Regular Plus	Premium	Diesel
Location I	1000	900	800	150
Location II	1800	1200	1100	250
Location III	800	1000	700	400

Find a matrix representing the total fuel sales at John's gas stations.

Solution The fuel sales can be represented by the matrices *A* (day 1) and *B* (day 2):

$$A = \begin{bmatrix} 1400 & 1200 & 1100 & 200 \\ 1600 & 900 & 1200 & 300 \\ 1200 & 1500 & 800 & 500 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1000 & 900 & 800 & 150 \\ 1800 & 1200 & 1100 & 250 \\ 800 & 1000 & 700 & 400 \end{bmatrix}$$

Note: Boldfaced words/characters enclosed in a box (for example, **Enter**) indicate that an action (click, select, or press) is required. Words/characters printed blue (for example, Chart sub-type:) indicate words/characters that appear on the screen. Words/characters printed in a typewriter font (for example, =(−2/3)*A2+2) indicate words/characters that need to be typed and entered.

We first enter the elements of the matrices A and B onto a spreadsheet. Next, we highlight the cells that will contain the matrix $A + B$, type =, highlight A, type +, highlight B, and then press ⎢**Ctrl-Shift-Enter**⎢. The resulting matrix $A + B$ is shown in Figure T4.

FIGURE **T4**
The matrix $A + B$

	A	B	C	D
23		A + B		
24	2400	2100	1900	350
25	3400	2100	2300	550
26	2000	2500	1500	900

TECHNOLOGY EXERCISES

Refer to the following matrices and perform the indicated operations.

$$A = \begin{bmatrix} 1.2 & 3.1 & -5.4 & 2.7 \\ 4.1 & 3.2 & 4.2 & -3.1 \\ 1.7 & 2.8 & -5.2 & 8.4 \end{bmatrix}$$

$$B = \begin{bmatrix} 6.2 & -3.2 & 1.4 & -1.2 \\ 3.1 & 2.7 & -1.2 & 1.7 \\ 1.2 & -1.4 & -1.7 & 2.8 \end{bmatrix}$$

1. $12.5A$

2. $-8.4B$

3. $A - B$

4. $B - A$

5. $1.3A + 2.4B$

6. $2.1A - 1.7B$

7. $3(A + B)$

8. $1.3(4.1A - 2.3B)$

2.5 Multiplication of Matrices

Matrix Product

In Section 2.4, we saw how matrices of the same size may be added or subtracted and how a matrix may be multiplied by a scalar (real number), an operation referred to as scalar multiplication. In this section we see how, with certain restrictions, one matrix may be multiplied by another matrix.

To define matrix multiplication, let's consider the following problem. On a certain day, Al's Service Station sold 1600 gallons of regular, 1000 gallons of regular plus, and 800 gallons of premium gasoline. If the price of gasoline on this day was $3.09 for regular, $3.29 for regular plus, and $3.45 for premium gasoline, find the total revenue realized by Al's for that day.

The day's sale of gasoline may be represented by the matrix

$$A = \begin{bmatrix} 1600 & 1000 & 800 \end{bmatrix} \quad \text{Row matrix (1 × 3)}$$

Next, we let the unit selling price of regular, regular plus, and premium gasoline be the entries in the matrix

$$B = \begin{bmatrix} 3.09 \\ 3.29 \\ 3.45 \end{bmatrix} \quad \text{Column matrix (3 × 1)}$$

The first entry in matrix A gives the number of gallons of regular gasoline sold, and the first entry in matrix B gives the selling price for each gallon of regular gasoline, so their product (1600)(3.09) gives the revenue realized from the sale of regular gaso-

line for the day. A similar interpretation of the second and third entries in the two matrices suggests that we multiply the corresponding entries to obtain the respective revenues realized from the sale of regular, regular plus, and premium gasoline. Finally, the total revenue realized by Al's from the sale of gasoline is given by adding these products to obtain

$$(1600)(3.09) + (1000)(3.29) + (800)(3.45) = 10,994$$

or \$10,994.

This example suggests that if we have a row matrix of size $1 \times n$,

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}$$

and a column matrix of size $n \times 1$,

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

then we may define the **matrix product** of A and B, written AB, by

$$AB = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n \qquad \textbf{(11)}$$

EXAMPLE 1 Let

$$A = \begin{bmatrix} 1 & -2 & 3 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & -2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix} = (1)(2) + (-2)(3) + (3)(0) + (5)(-1) = -9$$

APPLIED EXAMPLE 2 **Stock Transactions** Judy's stock holdings are given by the matrix

$$A = \begin{matrix} \text{GM} & \text{IBM} & \text{BAC} \\ \begin{bmatrix} 700 & 400 & 200 \end{bmatrix} \end{matrix}$$

At the close of trading on a certain day, the prices (in dollars per share) of these stocks are

$$B = \begin{matrix} \text{GM} \\ \text{IBM} \\ \text{BAC} \end{matrix} \begin{bmatrix} 50 \\ 120 \\ 42 \end{bmatrix}$$

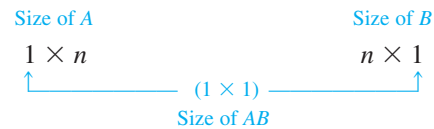What is the total value of Judy's holdings as of that day?
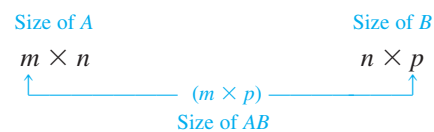
Solution   Judy's holdings are worth

$$AB = \begin{bmatrix} 700 & 400 & 200 \end{bmatrix} \begin{bmatrix} 50 \\ 120 \\ 42 \end{bmatrix} = (700)(50) + (400)(120) + (200)(42)$$

or \$91,400.   ■

Returning once again to the matrix product AB in Equation (11), observe that the number of columns of the row matrix A is *equal* to the number of rows of the column matrix B. Observe further that the product matrix AB has size 1×1 (a real number may be thought of as a 1×1 matrix). Schematically,

$$\underset{\text{Size of }A}{\underset{1 \times n}{\big\uparrow\underline{\hspace{2cm}}}} \underset{\text{Size of }AB}{\underset{(1 \times 1)}{\underline{\hspace{3cm}}}} \underset{\text{Size of }B}{\underset{n \times 1}{\underline{\hspace{2cm}}\big\uparrow}}$$

More generally, if A is a matrix of size $m \times n$ and B is a matrix of size $n \times p$ (the number of columns of A equals the numbers of rows of B), then the *matrix product* of A and B, AB, is defined and is a matrix of size $m \times p$. Schematically,

$$\underset{\text{Size of }A}{\underset{m \times n}{\big\uparrow\underline{\hspace{2cm}}}} \underset{\text{Size of }AB}{\underset{(m \times p)}{\underline{\hspace{3cm}}}} \underset{\text{Size of }B}{\underset{n \times p}{\underline{\hspace{2cm}}\big\uparrow}}$$

Next, let's illustrate the mechanics of matrix multiplication by computing the product of a 2×3 matrix A and a 3×4 matrix B. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$$

From the schematic

$$\text{Size of }A \quad \underset{(2 \times 4)}{\underset{\text{Size of }AB}{\underset{\big\uparrow\underline{\hspace{3cm}}}{2 \times 3 \overset{\overset{\text{Same}}{\downarrow\quad\quad\downarrow}}{\hspace{3cm}} 3 \times 4}}} \quad \text{Size of }B$$
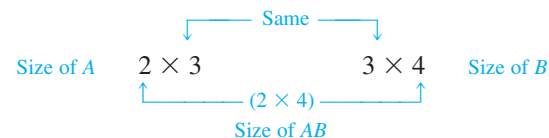
we see that the matrix product $C = AB$ is defined (since the number of columns of A equals the number of rows of B) and has size 2×4. Thus,

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}$$

The entries of C are computed as follows: The entry c_{11} (the entry in the *first* row, *first* column of C) is the product of the row matrix composed of the entries from the *first* row of A and the column matrix composed of the *first* column of B. Thus,

$$c_{11} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

The entry c_{12} (the entry in the *first* row, *second* column of C) is the product of the row matrix composed of the *first* row of A and the column matrix composed of the *second* column of B. Thus,

$$c_{12} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$

The other entries in C are computed in a similar manner.

EXAMPLE 3 Let

$$A = \begin{bmatrix} 3 & 1 & 4 \\ -1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 & -3 \\ 4 & -1 & 2 \\ 2 & 4 & 1 \end{bmatrix}$$

Compute AB.

Solution   The size of matrix A is 2×3, and the size of matrix B is 3×3. Since the number of columns of matrix A is equal to the number of rows of matrix B, the matrix product $C = AB$ is defined. Furthermore, the size of matrix C is 2×3. Thus,

$$\begin{bmatrix} 3 & 1 & 4 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 \\ 4 & -1 & 2 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

It remains now to determine the entries $c_{11}, c_{12}, c_{13}, c_{21}, c_{22}$, and c_{23}. We have

$$c_{11} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = (3)(1) + (1)(4) + (4)(2) = 15$$

$$c_{12} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = (3)(3) + (1)(-1) + (4)(4) = 24$$

$$c_{13} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = (3)(-3) + (1)(2) + (4)(1) = -3$$

$$c_{21} = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = (-1)(1) + (2)(4) + (3)(2) = 13$$

$$c_{22} = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = (-1)(3) + (2)(-1) + (3)(4) = 7$$

$$c_{23} = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = (-1)(-3) + (2)(2) + (3)(1) = 10$$

so the required product AB is given by

$$AB = \begin{bmatrix} 15 & 24 & -3 \\ 13 & 7 & 10 \end{bmatrix}$$

EXAMPLE 4 Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 2 & 3 \\ 3 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 3\cdot1 & +2\cdot2+1\cdot(-1) & 3\cdot3 & +2\cdot4+1\cdot2 & 3\cdot4 & +2\cdot1+1\cdot3 \\ (-1)\cdot1+2\cdot2+3\cdot(-1) & (-1)\cdot3+2\cdot4+3\cdot2 & (-1)\cdot4+2\cdot1+3\cdot3 \\ 3\cdot1 & +1\cdot2+4\cdot(-1) & 3\cdot3 & +1\cdot4+4\cdot2 & 3\cdot4 & +1\cdot1+4\cdot3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 19 & 17 \\ 0 & 11 & 7 \\ 1 & 21 & 25 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1\cdot3 & +3\cdot(-1)+4\cdot3 & 1\cdot2 & +3\cdot2+4\cdot1 & 1\cdot1 & +3\cdot3+4\cdot4 \\ 2\cdot3 & +4\cdot(-1)+1\cdot3 & 2\cdot2 & +4\cdot2+1\cdot1 & 2\cdot1 & +4\cdot3+1\cdot4 \\ (-1)\cdot3+2\cdot(-1)+3\cdot3 & (-1)\cdot2+2\cdot2+3\cdot1 & (-1)\cdot1+2\cdot3+3\cdot4 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 12 & 26 \\ 5 & 13 & 18 \\ 4 & 5 & 17 \end{bmatrix}$$ ▪

The preceding example shows that, in general, $AB \neq BA$ for two square matrices A and B. However, the following laws are valid for matrix multiplication.

Laws for Matrix Multiplication

If the products and sums are defined for the matrices A, B, and C, then

1. $(AB)C = A(BC)$ Associative law

2. $A(B + C) = AB + AC$ Distributive law

The square matrix of size n having 1s along the main diagonal and 0s elsewhere is called the identity matrix of size n.

Identity Matrix

The **identity matrix** of size n is given by

$$I_n = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix} \quad n \text{ rows}$$

n columns

The identity matrix has the properties that $I_n A = A$ for every $n \times r$ matrix A and $BI_n = B$ for every $s \times n$ matrix B. In particular, if A is a square matrix of size n, then

$$I_n A = A I_n = A$$

EXAMPLE 5 Let

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Then

$$I_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ -4 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ -4 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = A$$

$$A I_3 = \begin{bmatrix} 1 & 3 & 1 \\ -4 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ -4 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = A$$

so $I_3 A = A I_3 = A$, confirming our result for this special case. ∎

APPLIED EXAMPLE 6 Production Planning Ace Novelty received
an order from Magic World Amusement Park for 900 "Giant Pandas,"
1200 "Saint Bernards," and 2000 "Big Birds." Ace's management decided that
500 Giant Pandas, 800 Saint Bernards, and 1300 Big Birds could be manufac-
tured in their Los Angeles plant, and the balance of the order could be filled by
their Seattle plant. Each Panda requires 1.5 square yards of plush, 30 cubic feet of
stuffing, and 5 pieces of trim; each Saint Bernard requires 2 square yards of
plush, 35 cubic feet of stuffing, and 8 pieces of trim; and each Big Bird requires
2.5 square yards of plush, 25 cubic feet of stuffing, and 15 pieces of trim. The
plush costs \$4.50 per square yard, the stuffing costs 10 cents per cubic foot, and
the trim costs 25 cents per unit.

a. Find how much of each type of material must be purchased for each plant.
b. What is the total cost of materials incurred by each plant and the total cost of
materials incurred by Ace Novelty in filling the order?

Solution   The quantities of each type of stuffed animal to be produced at each
plant location may be expressed as a 2×3 *production matrix P*. Thus,

$$P = \begin{matrix} & \text{Pandas} & \text{St. Bernards} & \text{Birds} \\ \text{L.A.} & \\ \text{Seattle} \end{matrix} \begin{bmatrix} 500 & 800 & 1300 \\ 400 & 400 & 700 \end{bmatrix}$$

Similarly, we may represent the amount and type of material required to manu-
facture each type of animal by a 3×3 *activity matrix A*. Thus,

$$A = \begin{matrix} & \text{Plush} & \text{Stuffing} & \text{Trim} \\ \text{Pandas} & \\ \text{St. Bernards} & \\ \text{Birds} \end{matrix} \begin{bmatrix} 1.5 & 30 & 5 \\ 2 & 35 & 8 \\ 2.5 & 25 & 15 \end{bmatrix}$$

Finally, the unit cost for each type of material may be represented by the 3×1
cost matrix C.

$$C = \begin{matrix} \text{Plush} \\ \text{Stuffing} \\ \text{Trim} \end{matrix} \begin{bmatrix} 4.50 \\ 0.10 \\ 0.25 \end{bmatrix}$$

a. The amount of each type of material required for each plant is given by the
matrix *PA*. Thus,

$$PA = \begin{bmatrix} 500 & 800 & 1300 \\ 400 & 400 & 700 \end{bmatrix} \begin{bmatrix} 1.5 & 30 & 5 \\ 2 & 35 & 8 \\ 2.5 & 25 & 15 \end{bmatrix}$$

$$= \begin{matrix} \\ \text{L.A.} \\ \text{Seattle} \end{matrix} \begin{matrix} \text{Plush} & \text{Stuffing} & \text{Trim} \\ \begin{bmatrix} 5600 & 75{,}500 & 28{,}400 \\ 3150 & 43{,}500 & 15{,}700 \end{bmatrix} \end{matrix}$$

b. The total cost of materials for each plant is given by the matrix *PAC*:

$$PAC = \begin{bmatrix} 5600 & 75{,}500 & 28{,}400 \\ 3150 & 43{,}500 & 15{,}700 \end{bmatrix} \begin{bmatrix} 4.50 \\ 0.10 \\ 0.25 \end{bmatrix}$$

$$= \begin{matrix} \text{L.A.} \\ \text{Seattle} \end{matrix} \begin{bmatrix} 39{,}850 \\ 22{,}450 \end{bmatrix}$$

or \$39,850 for the L.A. plant and \$22,450 for the Seattle plant. Thus, the total cost of materials incurred by Ace Novelty is \$62,300.   ■

Matrix Representation

Example 7 shows how a system of linear equations may be written in a compact form with the help of matrices. (We will use this matrix equation representation in Section 2.6.)

EXAMPLE 7   Write the following system of linear equations in matrix form.

$$\begin{aligned} 2x - 4y + z &= 6 \\ -3x + 6y - 5z &= -1 \\ x - 3y + 7z &= 0 \end{aligned}$$

Solution   Let's write

$$A = \begin{bmatrix} 2 & -4 & 1 \\ -3 & 6 & -5 \\ 1 & -3 & 7 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix}$$

Note that *A* is just the 3×3 matrix of coefficients of the system, *X* is the 3×1 column matrix of unknowns (variables), and *B* is the 3×1 column matrix of constants. We now show that the required matrix representation of the system of linear equations is

$$AX = B$$

To see this, observe that

$$AX = \begin{bmatrix} 2 & -4 & 1 \\ -3 & 6 & -5 \\ 1 & -3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x - 4y + z \\ -3x + 6y - 5z \\ x - 3y + 7z \end{bmatrix}$$

Equating this 3×1 matrix with matrix *B* now gives

$$\begin{bmatrix} 2x - 4y + z \\ -3x + 6y - 5z \\ x - 3y + 7z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 0 \end{bmatrix}$$

which, by matrix equality, is easily seen to be equivalent to the given system of linear equations.   ■

2.5 Self-Check Exercises

1. Compute

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & 3 \\ 1 & 2 & -1 \end{bmatrix}$$

2. Write the following system of linear equations in matrix form:

$$\begin{aligned} y - 2z &= 1 \\ 2x - y + 3z &= 0 \\ x \quad\quad + 4z &= 7 \end{aligned}$$

3. On June 1, the stock holdings of Ash and Joan Robinson were given by the matrix

$$A = \begin{array}{c} \\ \text{Ash} \\ \text{Joan} \end{array} \begin{array}{cccc} \text{AT\&T} & \text{TWX} & \text{IBM} & \text{GM} \\ \begin{bmatrix} 2000 & 1000 & 500 & 5000 \\ 1000 & 2500 & 2000 & 0 \end{bmatrix} \end{array}$$

and the closing prices of AT&T, TWX, IBM, and GM were \$54, \$113, \$112, and \$70 per share, respectively. Use matrix multiplication to determine the separate values of Ash's and Joan's stock holdings as of that date.

Solutions to Self-Check Exercises 2.5 can be found on page 124.

2.5 Concept Questions

1. What is the difference between scalar multiplication and matrix multiplication? Give examples of each operation.

2. a. Suppose A and B are matrices whose products AB and BA are both defined. What can you say about the sizes of A and B?

b. If A, B, and C are matrices such that $A(B + C)$ is defined, what can you say about the relationship between the number of columns of A and the number of rows of C? Explain.

2.5 Exercises

In Exercises 1–4, the sizes of matrices A and B are given. Find the size of AB and BA whenever they are defined.

1. A is of size 2×3, and B is of size 3×5.

2. A is of size 3×4, and B is of size 4×3.

3. A is of size 1×7, and B is of size 7×1.

4. A is of size 4×4, and B is of size 4×4.

5. Let A be a matrix of size $m \times n$ and B be a matrix of size $s \times t$. Find conditions on m, n, s, and t such that both matrix products AB and BA are defined.

6. Find condition(s) on the size of a matrix A such that A^2 (that is, AA) is defined.

In Exercises 7–24, compute the indicated products.

7. $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

8. $\begin{bmatrix} -1 & 3 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 1 & 2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 2 & -1 \\ 4 & -1 & 0 \\ -5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$

11. $\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 0 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 3 \\ 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} -1 & 2 \\ 4 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 4 \end{bmatrix}$

15. $\begin{bmatrix} 0.1 & 0.9 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 1.2 & 0.4 \\ 0.5 & 2.1 \end{bmatrix}$

16. $\begin{bmatrix} 1.2 & 0.3 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.2 & 0.6 \\ 0.4 & -0.5 \end{bmatrix}$

17. $\begin{bmatrix} 6 & -3 & 0 \\ -2 & 1 & -8 \\ 4 & -4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 2 & 4 \\ -1 & -5 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 1 & 3 & -1 \end{bmatrix}$

19. $\begin{bmatrix} 3 & 0 & -2 & 1 \\ 1 & 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 2 \end{bmatrix}$

20. $\begin{bmatrix} 2 & 1 & -3 & 0 \\ 4 & -2 & -1 & 1 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 4 \\ 3 & -3 \\ 0 & -5 \end{bmatrix}$

21. $4\begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 1 & 4 & 0 \\ 0 & 1 & -2 \end{bmatrix}$

22. $3\begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 3 & -3 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

23. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & -3 & 2 \\ 7 & 1 & -5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

24. $2\begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 3 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -2 \\ 1 & 3 & 1 \end{bmatrix}$

In Exercises 25 and 26, let

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & -3 & 2 \\ -2 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & -3 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -1 & 2 \\ 3 & -2 & 1 \end{bmatrix}$$

25. Verify the validity of the associative law for matrix multiplication.

26. Verify the validity of the distributive law for matrix multiplication.

27. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

Compute AB and BA and hence deduce that matrix multiplication is, in general, not commutative.

28. Let

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 5 \\ 3 & -1 & -6 \\ 4 & 3 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & 5 & 6 \\ 3 & -1 & -6 \\ 2 & 2 & 3 \end{bmatrix}$$

a. Compute AB.
b. Compute AC.
c. Using the results of parts (a) and (b), conclude that $AB = AC$ does *not* imply that $B = C$.

29. Let

$$A = \begin{bmatrix} 3 & 0 \\ 8 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 4 & 5 \end{bmatrix}$$

Show that $AB = 0$, thereby demonstrating that for matrix multiplication the equation $AB = 0$ does not imply that one or both of the matrices A and B must be the zero matrix.

30. Let

$$A = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$$

Show that $A^2 = 0$. Compare this with the equation $a^2 = 0$, where a is a real number.

31. Find the matrix A such that

$$A\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 3 & 6 \end{bmatrix}$$

Hint: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

32. Let

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}$$

a. Compute $(A + B)^2$.
b. Compute $A^2 + 2AB + B^2$.
c. From the results of parts (a) and (b), show that in general $(A + B)^2 \neq A^2 + 2AB + B^2$.

33. Let

$$A = \begin{bmatrix} 2 & 4 \\ 5 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 8 \\ -7 & 3 \end{bmatrix}$$

a. Find A^T and show that $(A^T)^T = A$.
b. Show that $(A + B)^T = A^T + B^T$.
c. Show that $(AB)^T = B^T A^T$.

34. Let

$$A = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -4 \\ 2 & -2 \end{bmatrix}$$

a. Find A^T and show that $(A^T)^T = A$.
b. Show that $(A + B)^T = A^T + B^T$.
c. Show that $(AB)^T = B^T A^T$.

In Exercises 35–40, write the given system of linear equations in matrix form.

35. $2x - 3y = 7$
$3x - 4y = 8$

36. $2x \quad\;\; = 7$
$3x - 2y = 12$

37. $2x - 3y + 4z = 6$
$\quad\;\; 2y - 3z = 7$
$x - \;\; y + 2z = 4$

38. $\;\; x - 2y + 3z = -1$
$3x + 4y - 2z = \;\; 1$
$2x - 3y + 7z = \;\; 6$

39. $-x_1 + \;\; x_2 + \;\; x_3 = 0$
$2x_1 - \;\; x_2 - \;\; x_3 = 2$
$-3x_1 + 2x_2 + 4x_3 = 4$

40. $3x_1 - 5x_2 + 4x_3 = \;\; 10$
$4x_1 + 2x_2 - 3x_3 = -12$
$-x_1 \quad\quad\; + \;\; x_3 = \;\; -2$

41. INVESTMENTS William's and Michael's stock holdings are given by the matrix

$$A = \begin{matrix} \text{William} \\ \text{Michael} \end{matrix} \begin{array}{c} \begin{matrix} \text{BAC} & \text{GM} & \text{IBM} & \text{TRW} \end{matrix} \\ \begin{bmatrix} 200 & 300 & 100 & 200 \\ 100 & 200 & 400 & 0 \end{bmatrix} \end{array}$$

At the close of trading on a certain day, the prices (in dollars per share) of the stocks are given by the matrix

$$B = \begin{matrix} \text{BAC} \\ \text{GM} \\ \text{IBM} \\ \text{TRW} \end{matrix} \begin{bmatrix} 54 \\ 48 \\ 98 \\ 82 \end{bmatrix}$$

a. Find AB.
b. Explain the meaning of the entries in the matrix AB.

42. FOREIGN EXCHANGE Ethan just returned to the United States from a Southeast Asian trip and wishes to exchange the various foreign currencies that he has accumulated for U.S. dollars. He has 1200 Thai bahts, 80,000 Indonesian rupiahs, 42 Malaysian ringgits, and 36 Singapore dollars. Suppose the foreign exchange rates are U.S. $0.03 for one baht, U.S. $0.00011 for one rupiah, U.S. $0.294 for one Malaysian ringgit, and U.S. $0.656 for one Singapore dollar.
  a. Write a row matrix A giving the value of the various currencies that Ethan holds. (*Note:* The answer is *not* unique.)
  b. Write a column matrix B giving the exchange rates for the various currencies.
  c. If Ethan exchanges all of his foreign currencies for U.S. dollars, how many dollars will he have?

43. FOREIGN EXCHANGE Kaitlin and her friend Emma returned to the United States from a tour of four cities: Oslo, Stockholm, Copenhagen, and Saint Petersburg. They now wish to exchange the various foreign currencies that they have accumulated for U.S. dollars. Kaitlin has 82 Norwegian krones, 68 Swedish krones, 62 Danish krones, and 1200 Russian rubles. Emma has 64 Norwegian krones, 74 Swedish krones, 44 Danish krones, and 1600 Russian rubles. Suppose the exchange rates are U.S. $0.1651 for one Norwegian krone, U.S. $0.1462 for one Swedish krone, U.S. $0.1811 for one Danish krone, and U.S. $0.0387 for one Russian ruble.
  a. Write a 2×4 matrix A giving the values of the various foreign currencies held by Kaitlin and Emma. (*Note:* The answer is *not* unique.)
  b. Write a column matrix B giving the exchange rate for the various currencies.
  c. If both Kaitlin and Emma exchange all their foreign currencies for U.S. dollars, how many dollars will each have?

44. REAL ESTATE Bond Brothers, a real estate developer, builds houses in three states. The projected number of units of each model to be built in each state is given by the matrix

$$
\begin{array}{c}
 \\
\text{N.Y.} \\
A = \text{Conn.} \\
\text{Mass.}
\end{array}
\begin{array}{cccc}
\multicolumn{4}{c}{\text{Model}} \\
\text{I} & \text{II} & \text{III} & \text{IV} \\
\begin{bmatrix}
60 & 80 & 120 & 40 \\
20 & 30 & 60 & 10 \\
10 & 15 & 30 & 5
\end{bmatrix}
\end{array}
$$

The profits to be realized are $20,000, $22,000, $25,000, and $30,000, respectively, for each model I, II, III, and IV house sold.
  a. Write a column matrix B representing the profit for each type of house.
  b. Find the total profit Bond Brothers expects to earn in each state if all the houses are sold.

45. CHARITIES The amount of money raised by charity I, charity II, and charity III (in millions of dollars) in each of the years 2006, 2007, and 2008 is represented by the matrix A:

$$
\begin{array}{c}
 \\
2006 \\
A = 2007 \\
2008
\end{array}
\begin{array}{ccc}
\multicolumn{3}{c}{\text{Charity}} \\
\text{I} & \text{II} & \text{III} \\
\begin{bmatrix}
18.2 & 28.2 & 40.5 \\
19.6 & 28.6 & 42.6 \\
20.8 & 30.4 & 46.4
\end{bmatrix}
\end{array}
$$

On average, charity I puts 78% toward program cost, charity II puts 88% toward program cost, and charity III puts 80% toward program cost. Write a 3×1 matrix B reflecting the percentage put toward program cost by the charities. Then use matrix multiplication to find the total amount of money put toward program cost in each of the 3 yr by the charities under consideration.

46. BOX-OFFICE RECEIPTS The Cinema Center consists of four theaters: cinemas I, II, III, and IV. The admission price for one feature at the Center is $4 for children, $6 for students, and $8 for adults. The attendance for the Sunday matinee is given by the matrix

$$
\begin{array}{c}
 \\
\text{Cinema I} \\
\text{Cinema II} \\
A = \text{Cinema III} \\
\text{Cinema IV}
\end{array}
\begin{array}{ccc}
\text{Children} & \text{Students} & \text{Adults} \\
\begin{bmatrix}
225 & 110 & 50 \\
75 & 180 & 225 \\
280 & 85 & 110 \\
0 & 250 & 225
\end{bmatrix}
\end{array}
$$

Write a column vector B representing the admission prices. Then compute AB, the column vector showing the gross receipts for each theater. Finally, find the total revenue collected at the Cinema Center for admission that Sunday afternoon.

47. POLITICS: VOTER AFFILIATION Matrix A gives the percentage of eligible voters in the city of Newton, classified according to party affiliation and age group.

$$
\begin{array}{c}
 \\
\text{Under 30} \\
A = \text{30 to 50} \\
\text{Over 50}
\end{array}
\begin{array}{ccc}
\text{Dem.} & \text{Rep.} & \text{Ind.} \\
\begin{bmatrix}
0.50 & 0.30 & 0.20 \\
0.45 & 0.40 & 0.15 \\
0.40 & 0.50 & 0.10
\end{bmatrix}
\end{array}
$$

The population of eligible voters in the city by age group is given by the matrix B:

$$
\begin{array}{ccc}
\text{Under 30} & \text{30 to 50} & \text{Over 50} \\
B = [30,000 & 40,000 & 20,000]
\end{array}
$$

Find a matrix giving the total number of eligible voters in the city who will vote Democratic, Republican, and Independent.

48. 401(K) RETIREMENT PLANS Three network consultants, Alan, Maria, and Steven, each received a year-end bonus of $10,000, which they decided to invest in a 401(k) retirement plan sponsored by their employer. Under this plan, employees are allowed to place their investments in three funds: an equity index fund (I), a growth fund (II), and a global equity fund (III). The allocations of the investments (in dollars) of the three employees at the beginning of the year are summarized in the matrix

$$
\begin{array}{c}
\phantom{A=\text{Alan}} \quad \text{I} \qquad \text{II} \qquad \text{III} \\
A = \begin{array}{c} \text{Alan} \\ \text{Maria} \\ \text{Steven} \end{array}
\begin{bmatrix} 4000 & 3000 & 3000 \\ 2000 & 5000 & 3000 \\ 2000 & 3000 & 5000 \end{bmatrix}
\end{array}
$$

The returns of the three funds after 1 yr are given in the matrix

$$
B = \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array}
\begin{bmatrix} 0.18 \\ 0.24 \\ 0.12 \end{bmatrix}
$$

Which employee realized the best return on his or her investment for the year in question? The worst return?

49. COLLEGE ADMISSIONS A university admissions committee anticipates an enrollment of 8000 students in its freshman class next year. To satisfy admission quotas, incoming students have been categorized according to their sex and place of residence. The number of students in each category is given by the matrix

$$
\begin{array}{c}
\phantom{A=\text{In-state}} \quad \text{Male} \quad \text{Female} \\
A = \begin{array}{c} \text{In-state} \\ \text{Out-of-state} \\ \text{Foreign} \end{array}
\begin{bmatrix} 2700 & 3000 \\ 800 & 700 \\ 500 & 300 \end{bmatrix}
\end{array}
$$

By using data accumulated in previous years, the admissions committee has determined that these students will elect to enter the College of Letters and Science, the College of Fine Arts, the School of Business Administration, and the School of Engineering according to the percentages that appear in the following matrix:

$$
\begin{array}{c}
 \quad \text{L. \& S.} \quad \text{Fine Arts} \quad \text{Bus. Ad.} \quad \text{Eng.} \\
B = \begin{array}{c} \text{Male} \\ \text{Female} \end{array}
\begin{bmatrix} 0.25 & 0.20 & 0.30 & 0.25 \\ 0.30 & 0.35 & 0.25 & 0.10 \end{bmatrix}
\end{array}
$$

Find the matrix AB that shows the number of in-state, out-of-state, and foreign students expected to enter each discipline.

50. PRODUCTION PLANNING Refer to Example 6 in this section. Suppose Ace Novelty received an order from another amusement park for 1200 Pink Panthers, 1800 Giant Pandas, and 1400 Big Birds. The quantity of each type of stuffed animal to be produced at each plant is shown in the following production matrix:

$$
\begin{array}{c}
 \quad \text{Panthers} \quad \text{Pandas} \quad \text{Birds} \\
P = \begin{array}{c} \text{L.A.} \\ \text{Seattle} \end{array}
\begin{bmatrix} 700 & 1000 & 800 \\ 500 & 800 & 600 \end{bmatrix}
\end{array}
$$

Each Panther requires 1.3 yd^2 of plush, 20 ft^3 of stuffing, and 12 pieces of trim. Assume the materials required to produce the other two stuffed animals and the unit cost for each type of material are as given in Example 6.
  a. How much of each type of material must be purchased for each plant?
  b. What is the total cost of materials that will be incurred at each plant?

c. What is the total cost of materials incurred by Ace Novelty in filling the order?

51. COMPUTING PHONE BILLS Cindy regularly makes long-distance phone calls to three foreign cities—London, Tokyo, and Hong Kong. The matrices A and B give the lengths (in minutes) of her calls during peak and nonpeak hours, respectively, to each of these three cities during the month of June.

$$
\begin{array}{c}
\quad \text{London} \quad \text{Tokyo} \quad \text{Hong Kong} \\
A = \begin{bmatrix} 80 & 60 & 40 \end{bmatrix}
\end{array}
$$

and

$$
\begin{array}{c}
\quad \text{London} \quad \text{Tokyo} \quad \text{Hong Kong} \\
B = \begin{bmatrix} 300 & 150 & 250 \end{bmatrix}
\end{array}
$$

The costs for the calls (in dollars per minute) for the peak and nonpeak periods in the month in question are given, respectively, by the matrices

$$
C = \begin{array}{c} \text{London} \\ \text{Tokyo} \\ \text{Hong Kong} \end{array}
\begin{bmatrix} .34 \\ .42 \\ .48 \end{bmatrix}
\quad \text{and} \quad
D = \begin{array}{c} \text{London} \\ \text{Tokyo} \\ \text{Hong Kong} \end{array}
\begin{bmatrix} .24 \\ .31 \\ .35 \end{bmatrix}
$$

Compute the matrix $AC + BD$ and explain what it represents.

52. PRODUCTION PLANNING The total output of loudspeaker systems of the Acrosonic Company at their three production facilities for May and June is given by the matrices A and B, respectively, where

$$
\begin{array}{c}
\quad \begin{array}{cccc} \text{Model} & \text{Model} & \text{Model} & \text{Model} \\ \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\
A = \begin{array}{c} \text{Location I} \\ \text{Location II} \\ \text{Location III} \end{array}
\begin{bmatrix} 320 & 280 & 460 & 280 \\ 480 & 360 & 580 & 0 \\ 540 & 420 & 200 & 880 \end{bmatrix}
\end{array}
$$

$$
\begin{array}{c}
\quad \begin{array}{cccc} \text{Model} & \text{Model} & \text{Model} & \text{Model} \\ \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\
B = \begin{array}{c} \text{Location I} \\ \text{Location II} \\ \text{Location III} \end{array}
\begin{bmatrix} 210 & 180 & 330 & 180 \\ 400 & 300 & 450 & 40 \\ 420 & 280 & 180 & 740 \end{bmatrix}
\end{array}
$$

The unit production costs and selling prices for these loudspeakers are given by matrices C and D, respectively, where

$$
C = \begin{array}{c} \text{Model A} \\ \text{Model B} \\ \text{Model C} \\ \text{Model D} \end{array}
\begin{bmatrix} 120 \\ 180 \\ 260 \\ 500 \end{bmatrix}
\quad \text{and} \quad
D = \begin{array}{c} \text{Model A} \\ \text{Model B} \\ \text{Model C} \\ \text{Model D} \end{array}
\begin{bmatrix} 160 \\ 250 \\ 350 \\ 700 \end{bmatrix}
$$

Compute the following matrices and explain the meaning of the entries in each matrix.
  a. AC **b.** AD **c.** BC **d.** BD **e.** $(A + B)C$
  f. $(A + B)D$ **g.** $A(D - C)$
  h. $B(D - C)$ **i.** $(A + B)(D - C)$

53. DIET PLANNING A dietitian plans a meal around three foods. The number of units of vitamin A, vitamin C, and calcium in each ounce of these foods is represented by the matrix M, where

$$
\begin{array}{c}
\quad\quad\quad \text{Food I} \quad \text{Food II} \quad \text{Food III} \\
M = \begin{array}{c} \text{Vitamin A} \\ \text{Vitamin C} \\ \text{Calcium} \end{array}
\begin{bmatrix} 400 & 1200 & 800 \\ 110 & 570 & 340 \\ 90 & 30 & 60 \end{bmatrix}
\end{array}
$$

The matrices A and B represent the amount of each food (in ounces) consumed by a girl at two different meals, where

$$
\begin{array}{c}
\quad\quad \text{Food I} \quad \text{Food II} \quad \text{Food III} \\
A = \begin{bmatrix} 7 & 1 & 6 \end{bmatrix}
\end{array}
$$

$$
\begin{array}{c}
\quad\quad \text{Food I} \quad \text{Food II} \quad \text{Food III} \\
B = \begin{bmatrix} 9 & 3 & 2 \end{bmatrix}
\end{array}
$$

Calculate the following matrices and explain the meaning of the entries in each matrix.
a. MA^T **b.** MB^T **c.** $M(A + B)^T$

54. PRODUCTION PLANNING Hartman Lumber Company has two branches in the city. The sales of four of its products for the last year (in thousands of dollars) are represented by the matrix

$$
\begin{array}{c}
\quad\quad\quad\quad \text{Product} \\
\quad\quad\quad \text{A} \quad \text{B} \quad \text{C} \quad \text{D} \\
B = \begin{array}{c} \text{Branch I} \\ \text{Branch II} \end{array}
\begin{bmatrix} 5 & 2 & 8 & 10 \\ 3 & 4 & 6 & 8 \end{bmatrix}
\end{array}
$$

For the present year, management has projected that the sales of the four products in branch I will be 10% more than the corresponding sales for last year and the sales of the four products in branch II will be 15% more than the corresponding sales for last year.
a. Show that the sales of the four products in the two branches for the current year are given by the matrix AB, where

$$
A = \begin{bmatrix} 1.1 & 0 \\ 0 & 1.15 \end{bmatrix}
$$

Compute AB.
b. Hartman has m branches nationwide, and the sales of n of its products (in thousands of dollars) last year are represented by the matrix

$$
\begin{array}{c}
\quad\quad\quad\quad\quad\quad\quad \text{Product} \\
\quad\quad\quad\quad 1 \quad\quad 2 \quad\quad 3 \quad \cdots \quad n \\
B = \begin{array}{c} \text{Branch 1} \\ \text{Branch 2} \\ \vdots \\ \text{Branch } m \end{array}
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}
\end{array}
$$

Also, management has projected that the sales of the n products in branch 1, branch 2, . . . , branch m will be $r_1\%$, $r_2\%$, . . . , $r_m\%$, respectively, more than the corresponding sales for last year. Write the matrix A such that AB gives the sales of the n products in the m branches for the current year.

In Exercises 55–58, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

55. If A and B are matrices such that AB and BA are both defined, then A and B must be square matrices of the same order.

56. If A and B are matrices such that AB is defined and if c is a scalar, then $(cA)B = A(cB) = cAB$.

57. If A, B, and C are matrices and $A(B + C)$ is defined, then B must have the same size as C and the number of columns of A must be equal to the number of rows of B.

58. If A is a 2×4 matrix and B is a matrix such that ABA is defined, then the size of B must be 4×2.

2.5 Solutions to Self-Check Exercises

1. We compute

$$
\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix}
\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & 3 \\ 1 & 2 & -1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1(3) + 3(2) + 0(1) & 1(1) + 3(0) + 0(2) & 1(4) + 3(3) + 0(-1) \\ 2(3) + 4(2) - 1(1) & 2(1) + 4(0) - 1(2) & 2(4) + 4(3) - 1(-1) \end{bmatrix}
$$

$$
= \begin{bmatrix} 9 & 1 & 13 \\ 13 & 0 & 21 \end{bmatrix}
$$

2. Let

$$
A = \begin{bmatrix} 0 & 1 & -2 \\ 2 & -1 & 3 \\ 1 & 0 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}
$$

Then the given system may be written as the matrix equation

$$
AX = B
$$

3. Write

$$B = \begin{bmatrix} 54 \\ 113 \\ 112 \\ 70 \end{bmatrix} \begin{matrix} \text{AT\&T} \\ \text{TWX} \\ \text{IBM} \\ \text{GM} \end{matrix}$$

and compute the following:

$$AB = \begin{matrix} \text{Ash} \\ \text{Joan} \end{matrix} \begin{bmatrix} 2000 & 1000 & 500 & 5000 \\ 1000 & 2500 & 2000 & 0 \end{bmatrix} \begin{bmatrix} 54 \\ 113 \\ 112 \\ 70 \end{bmatrix}$$

$$= \begin{bmatrix} 627{,}000 \\ 560{,}500 \end{bmatrix} \begin{matrix} \text{Ash} \\ \text{Joan} \end{matrix}$$

We conclude that Ash's stock holdings were worth $627,000 and Joan's stock holdings were worth $560,500 on June 1.

USING TECHNOLOGY

Matrix Multiplication

Graphing Utility

A graphing utility can be used to perform matrix multiplication.

EXAMPLE 1 Let

$$A = \begin{bmatrix} 1.2 & 3.1 & -1.4 \\ 2.7 & 4.2 & 3.4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.8 & 1.2 & 3.7 \\ 6.2 & -0.4 & 3.3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.2 & 2.1 & 1.3 \\ 4.2 & -1.2 & 0.6 \\ 1.4 & 3.2 & 0.7 \end{bmatrix}$$

Find (a) AC and (b) $(1.1A + 2.3B)C$.

Solution First, we enter the matrices A, B, and C into the calculator.

a. Using matrix operations, we enter the expression $A*C$. We obtain the matrix

$$\begin{bmatrix} 12.5 & -5.68 & 2.44 \\ 25.64 & 11.51 & 8.41 \end{bmatrix}$$

(You may need to scroll the display on the screen to obtain the complete matrix.)

b. Using matrix operations, we enter the expression $(1.1A + 2.3B)C$. We obtain the matrix

$$\begin{bmatrix} 39.464 & 21.536 & 12.689 \\ 52.078 & 67.999 & 32.55 \end{bmatrix} \qquad \blacksquare$$

Excel

We use the **MMULT** function in Excel to perform matrix multiplication.

Note: Boldfaced words/characters in a box (for example, ⎡**Enter**⎤) indicate that an action (click, select, or press) is required. Words/characters printed blue (for example, Chart sub-type:) indicate words/characters that appear on the screen. Words/characters printed in a typewriter font (for example, =(−2/3)*A2+2) indicate words/characters that need to be typed and entered.

EXAMPLE 2 Let

$$A = \begin{bmatrix} 1.2 & 3.1 & -1.4 \\ 2.7 & 4.2 & 3.4 \end{bmatrix} \qquad B = \begin{bmatrix} 0.8 & 1.2 & 3.7 \\ 6.2 & -0.4 & 3.3 \end{bmatrix} \qquad C = \begin{bmatrix} 1.2 & 2.1 & 1.3 \\ 4.2 & -1.2 & 0.6 \\ 1.4 & 3.2 & 0.7 \end{bmatrix}$$

Find (a) AC and (b) $(1.1A + 2.3B)C$.

Solution

a. *First, enter the matrices A, B, and C onto a spreadsheet* (Figure T1).

	A	B	C	D	E	F	G
1		A				B	
2	1.2	3.1	−1.4		0.8	1.2	3.7
3	2.7	4.2	3.4		6.2	−0.4	3.3
4							
5		C					
6	1.2	2.1	1.3				
7	4.2	−1.2	0.6				
8	1.4	3.2	0.7				

FIGURE T1
Spreadsheet showing the matrices *A, B*, and *C*

Second, compute AC. Highlight the cells that will contain the matrix product *AC*, which has order 2×3. Type =MMULT(, highlight the cells in matrix *A*, type ,, highlight the cells in matrix *C*, type), and press **Ctrl-Shift-Enter**. The matrix product *AC* shown in Figure T2 will appear on your spreadsheet.

	A	B	C
10		AC	
11	12.5	−5.68	2.44
12	25.64	11.51	8.41

FIGURE T2
The matrix product *AC*

b. *Compute* $(1.1A + 2.3B)C$. Highlight the cells that will contain the matrix product $(1.1A + 2.3B)C$. Next, type =MMULT(1.1*, highlight the cells in matrix *A*, type +2.3*, highlight the cells in matrix *B*, type ,, highlight the cells in matrix *C*, type), and then press **Ctrl-Shift-Enter**. The matrix product shown in Figure T3 will appear on your spreadsheet.

	A	B	C
13		(1.1A+2.3B)C	
14	39.464	21.536	12.689
15	52.078	67.999	32.55

FIGURE T3
The matrix product $(1.1A + 2.3B)C$

TECHNOLOGY EXERCISES

In Exercises 1–8, refer to the following matrices and perform the indicated operations. Round your answers to two decimal places.

$$A = \begin{bmatrix} 1.2 & 3.1 & -1.2 & 4.3 \\ 7.2 & 6.3 & 1.8 & -2.1 \\ 0.8 & 3.2 & -1.3 & 2.8 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.7 & 0.3 & 1.2 & -0.8 \\ 1.2 & 1.7 & 3.5 & 4.2 \\ -3.3 & -1.2 & 4.2 & 3.2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.8 & 7.1 & 6.2 \\ 3.3 & -1.2 & 4.8 \\ 1.3 & 2.8 & -1.5 \\ 2.1 & 3.2 & -8.4 \end{bmatrix}$$

1. AC

2. CB

3. $(A + B)C$

4. $(2A + 3B)C$

5. $(2A - 3.1B)C$

6. $C(2.1A + 3.2B)$

7. $(4.1A + 2.7B)1.6C$

8. $2.5C(1.8A - 4.3B)$

In Exercises 9–12, refer to the following matrices and perform the indicated operations. Round your answers to two decimal places.

$$A = \begin{bmatrix} 2 & 5 & -4 & 2 & 8 \\ 6 & 7 & 2 & 9 & 6 \\ 4 & 5 & 4 & 4 & 4 \\ 9 & 6 & 8 & 3 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 6 & 7 & 5 \\ 3 & 4 & 6 & 2 \\ -5 & 8 & 4 & 3 \\ 8 & 6 & 9 & 5 \\ 4 & 7 & 8 & 8 \end{bmatrix}$$

$$C = \begin{bmatrix} 6.2 & 7.3 & -4.0 & 7.1 & 9.3 \\ 4.8 & 6.5 & 8.4 & -6.3 & 8.4 \\ 5.4 & 3.2 & 6.3 & 9.1 & -2.8 \\ 8.2 & 7.3 & 6.5 & 4.1 & 9.8 \\ 10.3 & 6.8 & 4.8 & -9.1 & 20.4 \end{bmatrix}$$

$$D = \begin{bmatrix} 4.6 & 3.9 & 8.4 & 6.1 & 9.8 \\ 2.4 & -6.8 & 7.9 & 11.4 & 2.9 \\ 7.1 & 9.4 & 6.3 & 5.7 & 4.2 \\ 3.4 & 6.1 & 5.3 & 8.4 & 6.3 \\ 7.1 & -4.2 & 3.9 & -6.4 & 7.1 \end{bmatrix}$$

9. Find AB and BA.

10. Find CD and DC. Is $CD = DC$?

11. Find $AC + AD$.

12. Find
   a. AC
   b. AD
   c. $A(C + D)$
   d. Is $A(C + D) = AC + AD$?

2.6   The Inverse of a Square Matrix

The Inverse of a Square Matrix

In this section, we discuss a procedure for finding the inverse of a matrix and show how the inverse can be used to help us solve a system of linear equations. The inverse of a matrix also plays a central role in the Leontief input–output model, which we discuss in Section 2.7.

Recall that if a is a nonzero real number, then there exists a unique real number a^{-1} $\left(\text{that is, } \frac{1}{a}\right)$ such that

$$a^{-1}a = \left(\frac{1}{a}\right)(a) = 1$$

The use of the (multiplicative) inverse of a real number enables us to solve algebraic equations of the form

$$ax = b \tag{12}$$

Multiplying both sides of (12) by a^{-1}, we have

$$a^{-1}(ax) = a^{-1}b$$
$$\left(\frac{1}{a}\right)(ax) = \frac{1}{a}(b)$$
$$x = \frac{b}{a}$$

For example, since the inverse of 2 is $2^{-1} = \frac{1}{2}$, we can solve the equation

$$2x = 5$$

by multiplying both sides of the equation by $2^{-1} = \frac{1}{2}$, giving

$$2^{-1}(2x) = 2^{-1} \cdot 5$$
$$x = \frac{5}{2}$$

We can use a similar procedure to solve the matrix equation

$$AX = B$$

where A, X, and B are matrices of the proper sizes. To do this we need the matrix equivalent of the inverse of a real number. Such a matrix, whenever it exists, is called the **inverse of a matrix.**

Inverse of a Matrix

Let A be a square matrix of size n. A square matrix A^{-1} of size n such that

$$A^{-1}A = AA^{-1} = I_n$$

is called the inverse of A.

Let's show that the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Explore & Discuss

In defining the inverse of a matrix A, why is it necessary to require that A be a square matrix?

has the matrix

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

as its inverse. Since

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

we see that A^{-1} is the inverse of A, as asserted.

Not every square matrix has an inverse. A square matrix that has an inverse is said to be **nonsingular.** A matrix that does not have an inverse is said to be **singular.** An example of a singular matrix is given by

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

If B had an inverse given by

$$B^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, and d are some appropriate numbers, then by the definition of an inverse we would have $BB^{-1} = I$; that is,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which implies that $0 = 1$—an impossibility! This contradiction shows that B does not have an inverse.

A Method for Finding the Inverse of a Square Matrix

The methods of Section 2.5 can be used to find the inverse of a nonsingular matrix. To discover such an algorithm, let's find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

Suppose A^{-1} exists and is given by

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, and d are to be determined. By the definition of an inverse, we have $AA^{-1} = I$; that is,

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} a + 2c & b + 2d \\ -a + 3c & -b + 3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But this matrix equation is equivalent to the two systems of linear equations

$$\left.\begin{array}{r} a + 2c = 1 \\ -a + 3c = 0 \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{r} b + 2d = 0 \\ -b + 3d = 1 \end{array}\right\}$$

with augmented matrices given by

$$\begin{bmatrix} 1 & 2 & | & 1 \\ -1 & 3 & | & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & | & 0 \\ -1 & 3 & | & 1 \end{bmatrix}$$

Note that the matrices of coefficients of the two systems are identical. This suggests that we solve the two systems of simultaneous linear equations by writing the following augmented matrix, which we obtain by joining the coefficient matrix and the two columns of constants:

$$\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ -1 & 3 & | & 0 & 1 \end{bmatrix}$$

Using the Gauss–Jordan elimination method, we obtain the following sequence of equivalent matrices:

$$\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ -1 & 3 & | & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 5 & | & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2}$$

$$\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & \frac{1}{5} & \frac{1}{5} \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & | & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & | & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

Thus, $a = \frac{3}{5}$, $b = -\frac{2}{5}$, $c = \frac{1}{5}$, and $d = \frac{1}{5}$, giving

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

The following computations verify that A^{-1} is indeed the inverse of A:

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

The preceding example suggests a general algorithm for computing the inverse of a square matrix of size n when it exists.

Finding the Inverse of a Matrix

Given the $n \times n$ matrix A:

1. Adjoin the $n \times n$ identity matrix I to obtain the augmented matrix

$$[A \mid I]$$

2. Use a sequence of row operations to reduce $[A \mid I]$ to the form

$$[I \mid B]$$

if possible.

Then the matrix B is the inverse of A.

Note   Although matrix multiplication is not generally commutative, it is possible to prove that if A has an inverse and $AB = I$, then $BA = I$ also. Hence, to verify that B is the inverse of A, it suffices to show that $AB = I$.   ■

EXAMPLE 1 Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Solution   We form the augmented matrix

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

and use the Gauss–Jordan elimination method to reduce it to the form $[I \mid B]$:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 1 & -1 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 + 3R_1 \\ R_3 + 2R_1}]{-R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 3 & -2 & 0 \\ 0 & -1 & 2 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{-R_2 \\ R_3 - R_2}]{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3]{R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

The inverse of A is the matrix

$$A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

We leave it to you to verify these results.   ■

Example 2 illustrates what happens to the reduction process when a matrix A does *not* have an inverse.

VIDEO

EXAMPLE 2 Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 3 & 5 \end{bmatrix}$$

Solution   We form the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 2 & 1 & 2 & | & 0 & 1 & 0 \\ 3 & 3 & 5 & | & 0 & 0 & 1 \end{bmatrix}$$

and use the Gauss–Jordan elimination method:

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 2 & 1 & 2 & | & 0 & 1 & 0 \\ 3 & 3 & 5 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & -3 & -4 & | & -2 & 1 & 0 \\ 0 & -3 & -4 & | & -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_2]{-R_2} \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 3 & 4 & | & 2 & -1 & 0 \\ 0 & 0 & 0 & | & -1 & -1 & 1 \end{bmatrix}$$

Since the entries in the last row of the 3×3 submatrix that comprises the left-hand side of the augmented matrix just obtained are all equal to zero, the latter cannot be reduced to the form $[I \mid B]$. Accordingly, we draw the conclusion that A is singular— that is, does not have an inverse.   ∎

More generally, we have the following criterion for determining when the inverse of a matrix does not exist.

Explore & Discuss

Explain in terms of solutions to systems of linear equations why the final augmented matrix in Example 2 implies that A has no inverse. *Hint:* See the discussion on pages 128–129.

Matrices That Have No Inverses

If there is a row to the left of the vertical line in the augmented matrix contain- ing all zeros, then the matrix does not have an inverse.

A Formula for the Inverse of a 2 × 2 Matrix

Before turning to some applications, we show an alternative method that employs a formula for finding the inverse of a 2×2 matrix. This method will prove useful in many situations; we will see an application in Example 5. The derivation of this for- mula is left as an exercise (Exercise 50).

Formula for the Inverse of a 2 × 2 Matrix

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Suppose $D = ad - bc$ is not equal to zero. Then A^{-1} exists and is given by

$$A^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{13}$$

Note   As an aid to memorizing the formula, note that D is the product of the elements along the main diagonal minus the product of the elements along the other diagonal:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad D = ad - bc$$

Main diagonal

Next, the matrix

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

is obtained by interchanging a and d and reversing the signs of b and c. Finally, A^{-1} is obtained by dividing this matrix by D.   ■

EXAMPLE 3 Find the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution   We first compute $D = (1)(4) - (2)(3) = 4 - 6 = -2$. Next, we write the matrix

$$\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Finally, dividing this matrix by D, we obtain

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$ ■

Solving Systems of Equations with Inverses

We now show how the inverse of a matrix may be used to solve certain systems of linear equations in which the number of equations in the system is equal to the number of variables. For simplicity, let's illustrate the process for a system of three linear equations in three variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad (14)$$

Let's write

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

You should verify that System (14) of linear equations may be written in the form of the matrix equation

$$AX = B \qquad (15)$$

If A is nonsingular, then the method of this section may be used to compute A^{-1}. Next, multiplying both sides of Equation (15) by A^{-1} (on the left), we obtain

$$A^{-1}AX = A^{-1}B \quad \text{or} \quad IX = A^{-1}B \quad \text{or} \quad X = A^{-1}B$$

the desired solution to the problem.

In the case of a system of n equations with n unknowns, we have the following more general result.

Using Inverses to Solve Systems of Equations

If $AX = B$ is a linear system of n equations in n unknowns and if A^{-1} exists, then

$$X = A^{-1}B$$

is the unique solution of the system.

The use of inverses to solve systems of equations is particularly advantageous when we are required to solve more than one system of equations, $AX = B$, involving the same coefficient matrix, A, and different matrices of constants, B. As you will see in Examples 4 and 5, we need to compute A^{-1} just once in each case.

EXAMPLE 4 Solve the following systems of linear equations:

a. $2x + y + z = 1$ **b.** $2x + y + z = 2$
$3x + 2y + z = 2$ $3x + 2y + z = -3$
$2x + y + 2z = -1$ $2x + y + 2z = 1$

Solution We may write the given systems of equations in the form

$$AX = B \quad \text{and} \quad AX = C$$

respectively, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad C = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

The inverse of the matrix A,

$$A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

was found in Example 1. Using this result, we find that the solution of the first system (a) is

$$X = A^{-1}B = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} (3)(1) + (-1)(2) + (-1)(-1) \\ (-4)(1) + (2)(2) + (1)(-1) \\ (-1)(1) + (0)(2) + (1)(-1) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

or $x = 2$, $y = -1$, and $z = -2$.

The solution of the second system (b) is

$$X = A^{-1}C = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -13 \\ -1 \end{bmatrix}$$

or $x = 8$, $y = -13$, and $z = -1$. ■

VIDEO

APPLIED EXAMPLE 5 **Capital Expenditure Planning** The management of Checkers Rent-A-Car plans to expand its fleet of rental cars for the next quarter by purchasing compact and full-size cars. The average cost of a compact car is $10,000, and the average cost of a full-size car is $24,000.

a. If a total of 800 cars is to be purchased with a budget of $12 million, how many cars of each size will be acquired?

b. If the predicted demand calls for a total purchase of 1000 cars with a budget of $14 million, how many cars of each type will be acquired?

Solution   Let x and y denote the number of compact and full-size cars to be purchased. Furthermore, let n denote the total number of cars to be acquired and b the amount of money budgeted for the purchase of these cars. Then,

$$x + \quad y = n$$
$$10{,}000x + 24{,}000y = b$$

This system of two equations in two variables may be written in the matrix form

$$AX = B$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 10{,}000 & 24{,}000 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad B = \begin{bmatrix} n \\ b \end{bmatrix}$$

Therefore,

$$X = A^{-1}B$$

Since A is a 2×2 matrix, its inverse may be found by using Formula (13). We find $D = (1)(24{,}000) - (1)(10{,}000) = 14{,}000$, so

$$A^{-1} = \frac{1}{14{,}000} \begin{bmatrix} 24{,}000 & -1 \\ -10{,}000 & 1 \end{bmatrix} = \begin{bmatrix} \frac{24{,}000}{14{,}000} & -\frac{1}{14{,}000} \\ -\frac{10{,}000}{14{,}000} & \frac{1}{14{,}000} \end{bmatrix}$$

Thus,

$$X = \begin{bmatrix} \frac{12}{7} & -\frac{1}{14{,}000} \\ -\frac{5}{7} & \frac{1}{14{,}000} \end{bmatrix} \begin{bmatrix} n \\ b \end{bmatrix}$$

a. Here, $n = 800$ and $b = 12{,}000{,}000$, so

$$X = A^{-1}B = \begin{bmatrix} \frac{12}{7} & -\frac{1}{14{,}000} \\ -\frac{5}{7} & \frac{1}{14{,}000} \end{bmatrix} \begin{bmatrix} 800 \\ 12{,}000{,}000 \end{bmatrix} = \begin{bmatrix} 514.3 \\ 285.7 \end{bmatrix}$$

Therefore, 514 compact cars and 286 full-size cars will be acquired in this case.

b. Here, $n = 1000$ and $b = 14{,}000{,}000$, so

$$X = A^{-1}B = \begin{bmatrix} \frac{12}{7} & -\frac{1}{14{,}000} \\ -\frac{5}{7} & \frac{1}{14{,}000} \end{bmatrix} \begin{bmatrix} 1000 \\ 14{,}000{,}000 \end{bmatrix} = \begin{bmatrix} 714.3 \\ 285.7 \end{bmatrix}$$

Therefore, 714 compact cars and 286 full-size cars will be purchased in this case. ■

2.6 Self-Check Exercises

1. Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -2 & 3 \end{bmatrix}$$

if it exists.

2. Solve the system of linear equations

$$2x + \quad y - \quad z = b_1$$
$$x + \quad y - \quad z = b_2$$
$$-x - 2y + 3z = b_3$$

where (a) $b_1 = 5$, $b_2 = 4$, $b_3 = -8$ and (b) $b_1 = 2$, $b_2 = 0$, $b_3 = 5$, by finding the inverse of the coefficient matrix.

3. Grand Canyon Tours offers air and ground scenic tours of the Grand Canyon. Tickets for the $7\frac{1}{2}$-hr tour cost \$169 for an adult and \$129 for a child, and each tour group is limited to 19 people. On three recent fully booked tours, total receipts were \$2931 for the first tour, \$3011 for the second tour, and \$2771 for the third tour. Determine how many adults and how many children were in each tour.

Solutions to Self-Check Exercises 2.6 can be found on page 138.

2.6 Concept Questions

1. What is the inverse of a matrix A?

2. Explain how you would find the inverse of a nonsingular matrix.

3. Give the formula for the inverse of the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

4. Explain how the inverse of a matrix can be used to solve a system of n linear equations in n unknowns. Can the method work for a system of m linear equations in n unknowns with $m \neq n$? Explain.

2.6 Exercises

In Exercises 1–4, show that the matrices are inverses of each other by showing that their product is the identity matrix *I*.

1. $\begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix}$ and $\begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 2 & 3 \\ 2 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} \\ 0 & 1 & -1 \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

4. $\begin{bmatrix} 2 & 4 & -2 \\ -4 & -6 & 1 \\ 3 & 5 & -1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & -3 & -4 \\ -\frac{1}{2} & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

In Exercises 5–16, find the inverse of the matrix, if it exists. Verify your answer.

5. $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

7. $\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$

9. $\begin{bmatrix} 2 & -3 & -4 \\ 0 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 4 & 2 & 2 \\ -1 & -3 & 4 \\ 3 & -1 & 6 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 & 0 \\ -3 & 4 & -2 \\ -5 & 0 & -2 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & -2 \\ -1 & 2 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 3 & -2 & 7 \\ -2 & 1 & 4 \\ 6 & -5 & 8 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 2 & -1 & -1 & 3 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 0 & -1 \\ 0 & 2 & -1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}$

In Exercises 17–24, (a) write a matrix equation that is equivalent to the system of linear equations and (b) solve the system using the inverses found in Exercises 5–16.

17. $2x + 5y = 3$
$x + 3y = 2$
(See Exercise 5.)

18. $2x + 3y = 5$
$3x + 5y = 8$
(See Exercise 6.)

19. $2x - 3y - 4z = 4$
$-z = 3$
$x - 2y + z = -8$
(See Exercise 9.)

20. $x_1 - x_2 + 3x_3 = 2$
$2x_1 + x_2 + 2x_3 = 2$
$-2x_1 - 2x_2 + x_3 = 3$
(See Exercise 10.)

21. $x + 4y - z = 3$
$2x + 3y - 2z = 1$
$-x + 2y + 3z = 7$
(See Exercise 13.)

22. $3x_1 - 2x_2 + 7x_3 = 6$
$-2x_1 + x_2 + 4x_3 = 4$
$6x_1 - 5x_2 + 8x_3 = 4$
(See Exercise 14.)

23. $x_1 + x_2 - x_3 + x_4 = 6$
$2x_1 + x_2 + x_3 = 4$
$2x_1 + x_2 + x_4 = 7$
$2x_1 - x_2 - x_3 + 3x_4 = 9$
(See Exercise 15.)

24. $x_1 + x_2 + 2x_3 + 3x_4 = 4$
$2x_1 + 3x_2 - x_4 = 11$
$2x_2 - x_3 + x_4 = 7$
$x_1 + 2x_2 + x_3 + x_4 = 6$
(See Exercise 16.)

In Exercises 25–32, (a) write each system of equations as a matrix equation and (b) solve the system of equations by using the inverse of the coefficient matrix.

25.
$$x + 2y = b_1$$
$$2x - y = b_2$$

where (i) $b_1 = 14, b_2 = 5$
and   (ii) $b_1 = 4, b_2 = -1$

26.
$$3x - 2y = b_1$$
$$4x + 3y = b_2$$

where (i) $b_1 = -6, b_2 = 10$
and   (ii) $b_1 = 3, b_2 = -2$

27.
$$x + 2y + z = b_1$$
$$x + y + z = b_2$$
$$3x + y + z = b_3$$

where (i) $b_1 = 7, b_2 = 4, b_3 = 2$
and   (ii) $b_1 = 5, b_2 = -3, b_3 = -1$

28.
$$x_1 + x_2 + x_3 = b_1$$
$$x_1 - x_2 + x_3 = b_2$$
$$x_1 - 2x_2 - x_3 = b_3$$

where (i) $b_1 = 5, b_2 = -3, b_3 = -1$
and   (ii) $b_1 = 1, b_2 = 4, b_3 = -2$

29.
$$3x + 2y - z = b_1$$
$$2x - 3y + z = b_2$$
$$x - y - z = b_3$$

where (i) $b_1 = 2, b_2 = -2, b_3 = 4$
and   (ii) $b_1 = 8, b_2 = -3, b_3 = 6$

30.
$$2x_1 + x_2 + x_3 = b_1$$
$$x_1 - 3x_2 + 4x_3 = b_2$$
$$-x_1 + x_3 = b_3$$

where (i) $b_1 = 1, b_2 = 4, b_3 = -3$
and   (ii) $b_1 = 2, b_2 = -5, b_3 = 0$

31.
$$x_1 + x_2 + x_3 + x_4 = b_1$$
$$x_1 - x_2 - x_3 + x_4 = b_2$$
$$x_2 + 2x_3 + 2x_4 = b_3$$
$$x_1 + 2x_2 + x_3 - 2x_4 = b_4$$

where (i) $b_1 = 1, b_2 = -1, b_3 = 4, b_4 = 0$
and   (ii) $b_1 = 2, b_2 = 8, b_3 = 4, b_4 = -1$

32.
$$x_1 + x_2 + 2x_3 + x_4 = b_1$$
$$4x_1 + 5x_2 + 9x_3 + x_4 = b_2$$
$$3x_1 + 4x_2 + 7x_3 + x_4 = b_3$$
$$2x_1 + 3x_2 + 4x_3 + 2x_4 = b_4$$

where (i) $b_1 = 3, b_2 = 6, b_3 = 5, b_4 = 7$
and   (ii) $b_1 = 1, b_2 = -1, b_3 = 0, b_4 = -4$

33. Let

$$A = \begin{bmatrix} 2 & 3 \\ -4 & -5 \end{bmatrix}$$

a. Find A^{-1}.   **b.** Show that $(A^{-1})^{-1} = A$.

34. Let

$$A = \begin{bmatrix} 6 & -4 \\ -4 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -5 \\ 4 & -7 \end{bmatrix}$$

a. Find AB, A^{-1}, and B^{-1}.
b. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

35. Let

$$A = \begin{bmatrix} 2 & -5 \\ 1 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$$

a. Find ABC, A^{-1}, B^{-1}, and C^{-1}.
b. Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

36. Find the matrix A if

$$\begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix} A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

37. Find the matrix A if

$$A \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$$

38. TICKET REVENUES Rainbow Harbor Cruises charges $16/adult and $8/child for a round-trip ticket. The records show that, on a certain weekend, 1000 people took the cruise on Saturday and 800 people took the cruise on Sunday. The total receipts for Saturday were $12,800 and the total receipts for Sunday were $9,600. Determine how many adults and children took the cruise on Saturday and on Sunday.

39. PRICING BelAir Publishing publishes a deluxe leather edition and a standard edition of its Daily Organizer. The company's marketing department estimates that x copies of the deluxe edition and y copies of the standard edition will be demanded per month when the unit prices are p dollars and q dollars, respectively, where x, y, p, and q are related by the following system of linear equations:

$$5x + y = 1000(70 - p)$$
$$x + 3y = 1000(40 - q)$$

Find the monthly demand for the deluxe edition and the standard edition when the unit prices are set according to the following schedules:
a. $p = 50$ and $q = 25$ **b.** $p = 45$ and $q = 25$
c. $p = 45$ and $q = 20$

40. NUTRITION/DIET PLANNING Bob, a nutritionist who works for the University Medical Center, has been asked to prepare special diets for two patients, Susan and Tom. Bob has decided that Susan's meals should contain at least 400 mg of calcium, 20 mg of iron, and 50 mg of vitamin C, whereas Tom's meals should contain at least 350 mg of calcium, 15 mg of iron, and 40 mg of vitamin C. Bob has also decided that the meals are to be prepared from three basic foods: food A, food B, and food C. The special nutritional contents of these foods are summarized in the accompanying table. Find how many ounces of each type of food should be used in a meal so that the minimum requirements of calcium, iron, and vitamin C are met for each patient's meals.

	Contents (mg/oz)		
	Calcium	**Iron**	**Vitamin C**
Food A	30	1	2
Food B	25	1	5
Food C	20	2	4

41. AGRICULTURE Jackson Farms has allotted a certain amount of land for cultivating soybeans, corn, and wheat. Cultivating 1 acre of soybeans requires 2 labor-hours, and cultivating 1 acre of corn or wheat requires 6 labor-hours. The cost of seeds for 1 acre of soybeans is $12, for 1 acre of corn is $20, and for 1 acre of wheat is $8. If all resources are to be used, how many acres of each crop should be cultivated if the following hold?
a. 1000 acres of land are allotted, 4400 labor-hours are available, and $13,200 is available for seeds.
b. 1200 acres of land are allotted, 5200 labor-hours are available, and $16,400 is available for seeds.

42. MIXTURE PROBLEM—FERTILIZER Lawnco produces three grades of commercial fertilizers. A 100-lb bag of grade A fertilizer contains 18 lb of nitrogen, 4 lb of phosphate, and 5 lb of potassium. A 100-lb bag of grade B fertilizer contains 20 lb of nitrogen and 4 lb each of phosphate and potassium. A 100-lb bag of grade C fertilizer contains 24 lb of nitrogen, 3 lb of phosphate, and 6 lb of potassium. How many 100-lb bags of each of the three grades of fertilizers should Lawnco produce if
a. 26,400 lb of nitrogen, 4900 lb of phosphate, and 6200 lb of potassium are available and all the nutrients are used?
b. 21,800 lb of nitrogen, 4200 lb of phosphate, and 5300 lb of potassium are available and all the nutrients are used?

43. INVESTMENT CLUBS A private investment club has a certain amount of money earmarked for investment in stocks. To arrive at an acceptable overall level of risk, the stocks that management is considering have been classified into three categories: high risk, medium risk, and low risk. Management estimates that high-risk stocks will have a rate of return of 15%/year; medium-risk stocks, 10%/year; and low-risk stocks, 6%/year. The members have decided that the investment in low-risk stocks should be equal to the sum of the investments in the stocks of the other two categories. Determine how much the club should invest in each type of stock in each of the following scenarios. (In all cases, assume that the entire sum available for investment is invested.)
a. The club has $200,000 to invest, and the investment goal is to have a return of $20,000/year on the total investment.
b. The club has $220,000 to invest, and the investment goal is to have a return of $22,000/year on the total investment.
c. The club has $240,000 to invest, and the investment goal is to have a return of $22,000/year on the total investment.

44. RESEARCH FUNDING The Carver Foundation funds three nonprofit organizations engaged in alternate-energy research

activities. From past data, the proportion of funds spent by each organization in research on solar energy, energy from harnessing the wind, and energy from the motion of ocean tides is given in the accompanying table.

	Proportion of Money Spent		
	Solar	**Wind**	**Tides**
Organization I	0.6	0.3	0.1
Organization II	0.4	0.3	0.3
Organization III	0.2	0.6	0.2

Find the amount awarded to each organization if the total amount spent by all three organizations on solar, wind, and tidal research is
a. $9.2 million, $9.6 million, and $5.2 million, respectively.
b. $8.2 million, $7.2 million, and $3.6 million, respectively.

45. Find the value(s) of k such that

$$A = \begin{bmatrix} 1 & 2 \\ k & 3 \end{bmatrix}$$

has an inverse. What is the inverse of A?
Hint: Use Formula 13.

46. Find the value(s) of k such that

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & k \\ -1 & 2 & k^2 \end{bmatrix}$$

has an inverse.
Hint: Find the value(s) of k such that the augmented matrix $[A \mid I]$ can be reduced to the form $[I \mid B]$.

In Exercises 47–49, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

47. If A is a square matrix with inverse A^{-1} and c is a nonzero real number, then

$$(cA)^{-1} = \left(\frac{1}{c}\right)A^{-1}$$

48. The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has an inverse if and only if $ad - bc = 0$.

49. If A^{-1} does not exist, then the system $AX = B$ of n linear equations in n unknowns does not have a unique solution.

50. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

a. Find A^{-1} if it exists.
b. Find a necessary condition for A to be nonsingular.
c. Verify that $AA^{-1} = A^{-1}A = I$.

2.6 Solutions to Self-Check Exercises

1. We form the augmented matrix

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -1 & -2 & 3 & 0 & 0 & 1 \end{array}\right]$$

and row-reduce as follows:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -1 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 \\ -1 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - 2R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 & 1 & 1 \end{array}\right] \xrightarrow[\substack{-R_2 \\ R_3 - R_2}]{R_1 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right] \xrightarrow{R_2 + R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 5 & 1 \\ 0 & 0 & 1 & -1 & 3 & 1 \end{array}\right]$$

From the preceding results, we see that

$$A^{-1} = \left[\begin{array}{ccc} 1 & -1 & 0 \\ -2 & 5 & 1 \\ -1 & 3 & 1 \end{array}\right]$$

2. a. We write the systems of linear equations in the matrix form

$$AX = B_1$$

where

$$A = \left[\begin{array}{ccc} 2 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -2 & 3 \end{array}\right] \qquad X = \left[\begin{array}{c} x \\ y \\ z \end{array}\right] \qquad B_1 = \left[\begin{array}{c} 5 \\ 4 \\ -8 \end{array}\right]$$

Now, using the results of Exercise 1, we have

$$X = \left[\begin{array}{c} x \\ y \\ z \end{array}\right] = A^{-1}B_1 = \left[\begin{array}{ccc} 1 & -1 & 0 \\ -2 & 5 & 1 \\ -1 & 3 & 1 \end{array}\right]\left[\begin{array}{c} 5 \\ 4 \\ -8 \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \\ -1 \end{array}\right]$$

Therefore, $x = 1$, $y = 2$, and $z = -1$.

b. Here A and X are as in part (a), but

$$B_2 = \left[\begin{array}{c} 2 \\ 0 \\ 5 \end{array}\right]$$

Therefore,

$$X = \left[\begin{array}{c} x \\ y \\ z \end{array}\right] = A^{-1}B_2 = \left[\begin{array}{ccc} 1 & -1 & 0 \\ -2 & 5 & 1 \\ -1 & 3 & 1 \end{array}\right]\left[\begin{array}{c} 2 \\ 0 \\ 5 \end{array}\right] = \left[\begin{array}{c} 2 \\ 1 \\ 3 \end{array}\right]$$

or $x = 2$, $y = 1$, and $z = 3$.

3. Let x denote the number of adults and y the number of children on a tour. Since the tours are filled to capacity, we have

$$x + y = 19$$

Next, since the total receipts for the first tour were \$2931 we have

$$169x + 129y = 2931$$

Therefore, the number of adults and the number of children in the first tour is found by solving the system of linear equations

$$\begin{aligned} x + y &= 19 \\ 169x + 129y &= 2931 \end{aligned} \qquad \textbf{(a)}$$

Similarly, we see that the number of adults and the number of children in the second and third tours are found by solving the systems

$$\begin{aligned} x + y &= 19 \\ 169x + 129y &= 3011 \end{aligned} \qquad \textbf{(b)}$$

$$\begin{aligned} x + y &= 19 \\ 169x + 129y &= 2771 \end{aligned} \qquad \textbf{(c)}$$

These systems may be written in the form

$$AX = B_1 \qquad AX = B_2 \qquad AX = B_3$$

where

$$A = \left[\begin{array}{cc} 1 & 1 \\ 169 & 129 \end{array}\right] \qquad X = \left[\begin{array}{c} x \\ y \end{array}\right]$$

$$B_1 = \left[\begin{array}{c} 19 \\ 2931 \end{array}\right] \qquad B_2 = \left[\begin{array}{c} 19 \\ 3011 \end{array}\right] \qquad B_3 = \left[\begin{array}{c} 19 \\ 2771 \end{array}\right]$$

To solve these systems, we first find A^{-1}. Using Formula (13), we obtain

$$A^{-1} = \left[\begin{array}{cc} -\frac{129}{40} & \frac{1}{40} \\ \frac{169}{40} & -\frac{1}{40} \end{array}\right]$$

Then, solving each system, we find

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B_1$$

$$= \begin{bmatrix} -\frac{129}{40} & \frac{1}{40} \\ \frac{169}{40} & -\frac{1}{40} \end{bmatrix} \begin{bmatrix} 19 \\ 2931 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix} \qquad \textbf{(a)}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B_2$$

$$= \begin{bmatrix} -\frac{129}{40} & \frac{1}{40} \\ \frac{169}{40} & -\frac{1}{40} \end{bmatrix} \begin{bmatrix} 19 \\ 3011 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ 5 \end{bmatrix} \qquad \textbf{(b)}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B_3$$

$$= \begin{bmatrix} -\frac{129}{40} & \frac{1}{40} \\ \frac{169}{40} & -\frac{1}{40} \end{bmatrix} \begin{bmatrix} 19 \\ 2771 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix} \qquad \textbf{(c)}$$

We conclude that there were
a. 12 adults and 7 children on the first tour.
b. 14 adults and 5 children on the second tour.
c. 8 adults and 11 children on the third tour.

USING TECHNOLOGY

Finding the Inverse of a Square Matrix

Graphing Utility

A graphing utility can be used to find the inverse of a square matrix.

EXAMPLE 1 Use a graphing utility to find the inverse of

$$\begin{bmatrix} 1 & 3 & 5 \\ -2 & 2 & 4 \\ 5 & 1 & 3 \end{bmatrix}$$

Solution   We first enter the given matrix as

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -2 & 2 & 4 \\ 5 & 1 & 3 \end{bmatrix}$$

Then, recalling the matrix A and using the $\boxed{x^{-1}}$ key, we find

$$A^{-1} = \begin{bmatrix} 0.1 & -0.2 & 0.1 \\ 1.3 & -1.1 & -0.7 \\ -0.6 & 0.7 & 0.4 \end{bmatrix}$$ ■

EXAMPLE 2 Use a graphing utility to solve the system

$$\begin{aligned} x + 3y + 5z &= 4 \\ -2x + 2y + 4z &= 3 \\ 5x + y + 3z &= 2 \end{aligned}$$

by using the inverse of the coefficient matrix.

Solution   The given system can be written in the matrix form $AX = B$, where

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -2 & 2 & 4 \\ 5 & 1 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

The solution is $X = A^{-1}B$. Entering the matrices A and B in the graphing utility and using the matrix multiplication capability of the utility gives the output shown in Figure T1—that is, $x = 0$, $y = 0.5$, and $z = 0.5$.

$$[A]^{-1} [B]$$

$$\begin{bmatrix} [0] \\ [.5] \\ [.5] \end{bmatrix}$$

Ans →
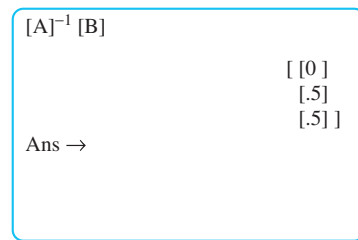
FIGURE **T1**
The TI-83/84 screen showing $A^{-1}B$

Excel

We use the function **MINVERSE** to find the inverse of a square matrix using Excel.

EXAMPLE 3 Find the inverse of

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -2 & 2 & 4 \\ 5 & 1 & 3 \end{bmatrix}$$

Solution

1. Enter the elements of matrix A onto a spreadsheet (Figure T2).
2. Compute the inverse of the matrix A: Highlight the cells that will contain the inverse matrix A^{-1}, type = MINVERSE(, highlight the cells containing matrix A, type), and press **Ctrl-Shift-Enter**. The desired matrix will appear in your spreadsheet (Figure T2).

	A	B	C
1		Matrix A	
2	1	3	5
3	−2	2	4
5	5	1	3
6			
7		Matrix A^{-1}	
8	0.1	−0.2	0.1
9	1.3	−1.1	−0.7
10	−0.6	0.7	0.4

FIGURE **T2**
Matrix A and its inverse, matrix A^{-1}

EXAMPLE 4 Solve the system

$$x + 3y + 5z = 4$$
$$-2x + 2y + 4z = 3$$
$$5x + y + 3z = 2$$

by using the inverse of the coefficient matrix.

**Solution** The given system can be written in the matrix form $AX = B$, where

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -2 & 2 & 4 \\ 5 & 1 & 3 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

The solution is $X = A^{-1}B$.

Note: Boldfaced words/characters enclosed in a box (for example, **Enter**) indicate that an action (click, select, or press) is required. Words/characters printed blue (for example, Chart sub-type:) indicate words/characters on the screen. Words/characters printed in a typewriter font (for example, = (−2/3) *A2+2) indicate words/characters that need to be typed and entered.

	A
12	Matrix X
13	5.55112E–17
14	0.5
15	0.5

FIGURE T3
Matrix *X* gives the solution to the problem.

1. *Enter the matrix B on a spreadsheet.*
2. *Compute $A^{-1}B$.* Highlight the cells that will contain the matrix *X*, and then type `=MMULT(`, highlight the cells in the matrix A^{-1}, type `,`, highlight the cells in the matrix *B*, type `)`, and press $\boxed{\textbf{Ctrl-Shift-Enter}}$. (*Note*: The matrix A^{-1} was found in Example 3.) The matrix *X* shown in Figure T3 will appear on your spreadsheet. Thus, $x = 0$, $y = 0.5$, and $z = 0.5$. ∎

TECHNOLOGY EXERCISES

In Exercises 1–6, find the inverse of the matrix. Round your answers to two decimal places.

1. $\begin{bmatrix} 1.2 & 3.1 & -2.1 \\ 3.4 & 2.6 & 7.3 \\ -1.2 & 3.4 & -1.3 \end{bmatrix}$

2. $\begin{bmatrix} 4.2 & 3.7 & 4.6 \\ 2.1 & -1.3 & -2.3 \\ 1.8 & 7.6 & -2.3 \end{bmatrix}$

3. $\begin{bmatrix} 1.1 & 2.3 & 3.1 & 4.2 \\ 1.6 & 3.2 & 1.8 & 2.9 \\ 4.2 & 1.6 & 1.4 & 3.2 \\ 1.6 & 2.1 & 2.8 & 7.2 \end{bmatrix}$

4. $\begin{bmatrix} 2.1 & 3.2 & -1.4 & -3.2 \\ 6.2 & 7.3 & 8.4 & 1.6 \\ 2.3 & 7.1 & 2.4 & -1.3 \\ -2.1 & 3.1 & 4.6 & 3.7 \end{bmatrix}$

5. $\begin{bmatrix} 2 & -1 & 3 & 2 & 4 \\ 3 & 2 & -1 & 4 & 1 \\ 3 & 2 & 6 & 4 & -1 \\ 2 & 1 & -1 & 4 & 2 \\ 3 & 4 & 2 & 5 & 6 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 4 & 2 & 3 & 1.4 \\ 6 & 2.4 & 5 & 1.2 & 3 \\ 4 & 1 & 2 & 3 & 1.2 \\ -1 & 2 & -3 & 4 & 2 \\ 1.1 & 2.2 & 3 & 5.1 & 4 \end{bmatrix}$

In Exercises 7–10, solve the system of linear equations by first writing the system in the form $AX = B$ and then solving the resulting system by using A^{-1}. Round your answers to two decimal places.

7. $\begin{aligned} 2x - 3y + 4z &= 2.4 \\ 3x + 2y - 7z &= -8.1 \\ x + 4y - 2z &= 10.2 \end{aligned}$

8. $\begin{aligned} 3.2x - 4.7y + 3.2z &= 7.1 \\ 2.1x + 2.6y + 6.2z &= 8.2 \\ 5.1x - 3.1y - 2.6z &= -6.5 \end{aligned}$

9. $\begin{aligned} 3x_1 - 2x_2 + 4x_3 - 8x_4 &= 8 \\ 2x_1 + 3x_2 - 2x_3 + 6x_4 &= 4 \\ 3x_1 + 2x_2 - 6x_3 - 7x_4 &= -2 \\ 4x_1 - 7x_2 + 4x_3 + 6x_4 &= 22 \end{aligned}$

10. $\begin{aligned} 1.2x_1 + 2.1x_2 - 3.2x_3 + 4.6x_4 &= 6.2 \\ 3.1x_1 - 1.2x_2 + 4.1x_3 - 3.6x_4 &= -2.2 \\ 1.8x_1 + 3.1x_2 - 2.4x_3 + 8.1x_4 &= 6.2 \\ 2.6x_1 - 2.4x_2 + 3.6x_3 - 4.6x_4 &= 3.6 \end{aligned}$

2.7 Leontief Input–Output Model

Input–Output Analysis

One of the many important applications of matrix theory to the field of economics is the study of the relationship between industrial production and consumer demand. At the heart of this analysis is the Leontief input–output model pioneered by Wassily Leontief, who was awarded a Nobel Prize in economics in 1973 for his contributions to the field.

To illustrate this concept, let's consider an oversimplified economy consisting of three sectors: agriculture (A), manufacturing (M), and service (S). In general, part of the output of one sector is absorbed by another sector through interindustry purchases, with the excess available to fulfill consumer demands. The relationship governing both intraindustrial and interindustrial sales and purchases is conveniently represented by means of an **input–output matrix:**

$$
\begin{array}{c}
\text{Input} \\
\text{(amount used in production)}
\end{array}
\quad
\begin{array}{cc}
 & \text{Output (amount produced)} \\
 & \begin{array}{ccc} A & M & S \end{array} \\
\begin{array}{c} A \\ M \\ S \end{array} &
\begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}
\end{array}
\qquad (16)
$$

The first column (read from top to bottom) tells us that the production of 1 unit of agricultural products requires the consumption of 0.2 unit of agricultural products, 0.2 unit of manufactured goods, and 0.1 unit of services. The second column tells us that the production of 1 unit of manufactured goods requires the consumption of 0.2 unit of agricultural products, 0.4 unit of manufactured goods, and 0.2 unit of services. Finally, the third column tells us that the production of 1 unit of services requires the consumption of 0.1 unit each of agricultural products and manufactured goods and 0.3 unit of services.

APPLIED EXAMPLE 1 Input–Output Analysis Refer to the input–output matrix (16).

a. If the units are measured in millions of dollars, determine the amount of agricultural products consumed in the production of $100 million worth of manufactured goods.
b. Determine the dollar amount of manufactured goods required to produce $200 million worth of all goods and services in the economy.

Solution

a. The production of 1 unit requires the consumption of 0.2 unit of agricultural products. Thus, the amount of agricultural products consumed in the production of $100 million worth of manufactured goods is given by $(100)(0.2)$, or $20 million.
b. The amount of manufactured goods required to produce 1 unit of all goods and services in the economy is given by adding the numbers of the second row of the input–output matrix—that is, $0.2 + 0.4 + 0.1$, or 0.7 unit. Therefore, the production of $200 million worth of all goods and services in the economy requires $200(0.7)$ million, or $140 million, worth of manufactured goods. ■

Next, suppose the total output of goods of the agriculture and manufacturing sectors and the total output from the service sector of the economy are given by x, y, and z units, respectively. What is the value of agricultural products consumed in the internal process of producing this total output of various goods and services?

To answer this question, we first note, by examining the input–output matrix

$$
\begin{array}{c}
\text{Input} \\

\end{array}
\quad
\begin{array}{cc}
 & \text{Output} \\
 & \begin{array}{ccc} A & M & S \end{array} \\
\begin{array}{c} A \\ M \\ S \end{array} &
\begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}
\end{array}
$$

that 0.2 unit of agricultural products is required to produce 1 unit of agricultural products, so the amount of agricultural goods required to produce x units of agricultural products is given by $0.2x$ unit. Next, again referring to the input–output matrix, we see that 0.2 unit of agricultural products is required to produce 1 unit of manufactured goods, so the requirement for producing y units of the latter is $0.2y$ unit of agricultural products. Finally, we see that 0.1 unit of agricultural goods is required to produce 1 unit of services, so the value of agricultural products required to produce z units of services is $0.1z$ unit. Thus, the total amount of agricultural products required to produce the total output of goods and services in the economy is

$$0.2x + 0.2y + 0.1z$$

units. In a similar manner, we see that the total amount of manufactured goods and the total value of services required to produce the total output of goods and services in the economy are given by

$$0.2x + 0.4y + 0.1z$$
$$0.1x + 0.2y + 0.3z$$

respectively.

These results could also be obtained using matrix multiplication. To see this, write the total output of goods and services x, y, and z as a 3×1 matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{Total output matrix}$$

The matrix X is called the **total output matrix.** Letting A denote the input–output matrix, we have

$$A = \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} \qquad \text{Input–output matrix}$$

Then, the product

$$AX = \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 0.2x + 0.2y + 0.1z \\ 0.2x + 0.4y + 0.1z \\ 0.1x + 0.2y + 0.3z \end{bmatrix} \qquad \text{Internal consumption matrix}$$

is a 3×1 matrix whose entries represent the respective values of the agricultural products, manufactured goods, and services consumed in the internal process of production. The matrix AX is referred to as the **internal consumption matrix.**

Now, since X gives the total production of goods and services in the economy, and AX, as we have just seen, gives the amount of goods and services consumed in the production of these goods and services, it follows that the 3×1 matrix $X - AX$ gives the net output of goods and services that is exactly enough to satisfy consumer demands. Letting matrix D represent these consumer demands, we are led to the following matrix equation:

$$X - AX = D$$
$$(I - A)X = D$$

where I is the 3×3 identity matrix.

Assuming that the inverse of $(I - A)$ exists, multiplying both sides of the last equation by $(I - A)^{-1}$ on the left yields

$$X = (I - A)^{-1}D$$

Leontief Input–Output Model

In a **Leontief input–output model**, the matrix equation giving the net output of goods and services needed to satisfy consumer demand is

$$\underset{\substack{\text{Total} \\ \text{output}}}{X} \; - \; \underset{\substack{\text{Internal} \\ \text{consumption}}}{AX} \; = \; \underset{\substack{\text{Consumer} \\ \text{demand}}}{D}$$

where X is the total output matrix, A is the input–output matrix, and D is the matrix representing consumer demand.

The solution to this equation is

$$X = (I - A)^{-1}D \qquad \text{Assuming that } (I - A)^{-1} \text{ exists} \tag{17}$$

which gives the amount of goods and services that must be produced to satisfy consumer demand.

Equation (17) gives us a means of finding the amount of goods and services to be produced in order to satisfy a given level of consumer demand, as illustrated by the following example.

APPLIED EXAMPLE 2 Input–Output Model for a Three-Sector Economy For the three-sector economy with input–output matrix given by (16), which is reproduced here:

$$A = \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} \qquad \text{Each unit equals \$1 million.}$$

a. Find the total output of goods and services needed to satisfy a consumer demand of $100 million worth of agricultural products, $80 million worth of manufactured goods, and $50 million worth of services.

b. Find the value of the goods and services consumed in the internal process of production in order to meet this total output.

Solution

a. We are required to determine the total output matrix

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where x, y, and z denote the value of the agricultural products, the manufactured goods, and services, respectively. The matrix representing the consumer demand is given by

$$D = \begin{bmatrix} 100 \\ 80 \\ 50 \end{bmatrix}$$

Next, we compute

$$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.2 & 0.1 \\ 0.2 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.2 & -0.1 \\ -0.2 & 0.6 & -0.1 \\ -0.1 & -0.2 & 0.7 \end{bmatrix}$$

Using the method of Section 2.6, we find (to two decimal places)

$$(I - A)^{-1} = \begin{bmatrix} 1.43 & 0.57 & 0.29 \\ 0.54 & 1.96 & 0.36 \\ 0.36 & 0.64 & 1.57 \end{bmatrix}$$

Finally, using Equation (17), we find

$$X = (I - A)^{-1}D = \begin{bmatrix} 1.43 & 0.57 & 0.29 \\ 0.54 & 1.96 & 0.36 \\ 0.36 & 0.64 & 1.57 \end{bmatrix}\begin{bmatrix} 100 \\ 80 \\ 50 \end{bmatrix} = \begin{bmatrix} 203.1 \\ 228.8 \\ 165.7 \end{bmatrix}$$

To fulfill consumer demand, $203 million worth of agricultural products, $229 million worth of manufactured goods, and $166 million worth of services should be produced.

b. The amount of goods and services consumed in the internal process of production is given by AX or, equivalently, by $X - D$. In this case it is more convenient to use the latter, which gives the required result of

$$\begin{bmatrix} 203.1 \\ 228.8 \\ 165.7 \end{bmatrix} - \begin{bmatrix} 100 \\ 80 \\ 50 \end{bmatrix} = \begin{bmatrix} 103.1 \\ 148.8 \\ 115.7 \end{bmatrix}$$

or $103 million worth of agricultural products, $149 million worth of manufactured goods, and $116 million worth of services. ■

APPLIED EXAMPLE 3 An Input–Output Model for a Three-Product Company TKK Corporation, a large conglomerate, has three subsidiaries engaged in producing raw rubber, manufacturing tires, and manufacturing other rubber-based goods. The production of 1 unit of raw rubber requires the consumption of 0.08 unit of rubber, 0.04 unit of tires, and 0.02 unit of other rubber-based goods. To produce 1 unit of tires requires 0.6 unit of raw rubber, 0.02 unit of tires, and 0 unit of other rubber-based goods. To produce 1 unit of other rubber-based goods requires 0.3 unit of raw rubber, 0.01 unit of tires, and 0.06 unit of other rubber-based goods. Market research indicates that the demand for the following year will be $200 million for raw rubber, $800 million for tires, and $120 million for other rubber-based products. Find the level of production for each subsidiary in order to satisfy this demand.

Solution   View the corporation as an economy having three sectors and with an input–output matrix given by

$$A = \begin{matrix} \\ \text{Raw rubber} \\ \text{Tires} \\ \text{Goods} \end{matrix} \begin{matrix} \text{Raw} \\ \text{rubber} \quad \text{Tires} \quad \text{Goods} \\ \begin{bmatrix} 0.08 & 0.60 & 0.30 \\ 0.04 & 0.02 & 0.01 \\ 0.02 & 0 & 0.06 \end{bmatrix} \end{matrix}$$

Using Equation (17), we find that the required level of production is given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (I - A)^{-1}D$$

where x, y, and z denote the outputs of raw rubber, tires, and other rubber-based goods and where

$$D = \begin{bmatrix} 200 \\ 800 \\ 120 \end{bmatrix}$$

Now,

$$I - A = \begin{bmatrix} 0.92 & -0.60 & -0.30 \\ -0.04 & 0.98 & -0.01 \\ -0.02 & 0 & 0.94 \end{bmatrix}$$

You are asked to verify that

$$(I - A)^{-1} = \begin{bmatrix} 1.12 & 0.69 & 0.37 \\ 0.05 & 1.05 & 0.03 \\ 0.02 & 0.01 & 1.07 \end{bmatrix} \quad \textit{See Exercise 7.}$$

Therefore,

$$X = (I - A)^{-1}D = \begin{bmatrix} 1.12 & 0.69 & 0.37 \\ 0.05 & 1.05 & 0.03 \\ 0.02 & 0.01 & 1.07 \end{bmatrix} \begin{bmatrix} 200 \\ 800 \\ 120 \end{bmatrix} = \begin{bmatrix} 820.4 \\ 853.6 \\ 140.4 \end{bmatrix}$$

To fulfill the predicted demand, $820 million worth of raw rubber, $854 million worth of tires, and $140 million worth of other rubber-based goods should be produced. ■

2.7 Self-Check Exercises

1. Solve the matrix equation $(I - A)X = D$ for x and y, given that

$$A = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix} \qquad D = \begin{bmatrix} 50 \\ 10 \end{bmatrix}$$

2. A simple economy consists of two sectors: agriculture (A) and transportation (T). The input–output matrix for this economy is given by

$$A = \begin{matrix} & \begin{matrix} A & & T \end{matrix} \\ \begin{matrix} A \\ T \end{matrix} & \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} \end{matrix}$$

a. Find the gross output of agricultural products needed to satisfy a consumer demand for $50 million worth of agricultural products and $10 million worth of transportation.

b. Find the value of agricultural products and transportation consumed in the internal process of production in order to meet the gross output.

Solutions to Self-Check Exercises 2.7 can be found on page 148.

2.7 Concept Questions

1. What do the quantities X, AX, and D represent in the matrix equation $X - AX = D$ for a Leontief input–output model?

2. What is the solution to the matrix equation $X - AX = D$? Does the solution to this equation always exist? Why or why not?

2.7 Exercises

1. AN INPUT–OUTPUT MATRIX FOR A THREE-SECTOR ECONOMY
A simple economy consists of three sectors: agriculture
(A), manufacturing (M), and transportation (T). The input–
output matrix for this economy is given by

$$
\begin{array}{c}
\quad A\quad\; M\quad\; T \\
\begin{array}{c}A\\M\\T\end{array}
\begin{bmatrix}
0.4 & 0.1 & 0.1 \\
0.1 & 0.4 & 0.3 \\
0.2 & 0.2 & 0.2
\end{bmatrix}
\end{array}
$$

a. Determine the amount of agricultural products con-
sumed in the production of $100 million worth of man-
ufactured goods.
b. Determine the dollar amount of manufactured goods
required to produce $200 million worth of all goods in
the economy.
c. Which sector consumes the greatest amount of agricul-
tural products in the production of a unit of goods in that
sector? The least?

2. AN INPUT–OUTPUT MATRIX FOR A FOUR-SECTOR ECONOMY
The relationship governing the intraindustrial and inter-
industrial sales and purchases of four basic industries—
agriculture (A), manufacturing (M), transportation (T), and
energy (E)—of a certain economy is given by the follow-
ing input–output matrix.

$$
\begin{array}{c}
\quad A\quad\; M\quad\; T\quad\; E \\
\begin{array}{c}A\\M\\T\\E\end{array}
\begin{bmatrix}
0.3 & 0.2 & 0 & 0.1 \\
0.2 & 0.3 & 0.2 & 0.1 \\
0.2 & 0.2 & 0.1 & 0.3 \\
0.1 & 0.2 & 0.3 & 0.2
\end{bmatrix}
\end{array}
$$

a. How many units of energy are required to produce 1
unit of manufactured goods?
b. How many units of energy are required to produce 3
units of all goods in the economy?
c. Which sector of the economy is least dependent on the
cost of energy?
d. Which sector of the economy has the smallest intrain-
dustry purchases (sales)?

**In Exercises 3–6, use the input–output matrix A and the
consumer demand matrix D to solve the matrix equation
(I − A)X = D for the total output matrix X.**

3. $A = \begin{bmatrix} 0.4 & 0.2 \\ 0.3 & 0.1 \end{bmatrix}$ and $D = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$

4. $A = \begin{bmatrix} 0.2 & 0.3 \\ 0.5 & 0.2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

5. $A = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.5 \end{bmatrix}$ and $D = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$

6. $A = \begin{bmatrix} 0.6 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}$ and $D = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$

7. Let

$$
A = \begin{bmatrix}
0.08 & 0.60 & 0.30 \\
0.04 & 0.02 & 0.01 \\
0.02 & 0 & 0.06
\end{bmatrix}
$$

Show that

$$
(I - A)^{-1} = \begin{bmatrix}
1.13 & 0.69 & 0.37 \\
0.05 & 1.05 & 0.03 \\
0.02 & 0.02 & 1.07
\end{bmatrix}
$$

8. AN INPUT–OUTPUT MODEL FOR A TWO-SECTOR ECONOMY
A simple economy consists of two industries: agriculture
and manufacturing. The production of 1 unit of agricultural
products requires the consumption of 0.2 unit of agricul-
tural products and 0.3 unit of manufactured goods. The
production of 1 unit of manufactured goods requires the
consumption of 0.4 unit of agricultural products and
0.3 unit of manufactured goods.
a. Find the total output of goods needed to satisfy a
consumer demand for $100 million worth of agricul-
tural products and $150 million worth of manufactured
goods.
b. Find the value of the goods consumed in the internal
process of production in order to meet the gross output.

9. Rework Exercise 8 if the consumer demand for the output
of agricultural products and the consumer demand for
manufactured goods are $120 million and $140 million,
respectively.

10. Refer to Example 3. Suppose the demand for raw rubber
increases by 10%, the demand for tires increases by 20%,
and the demand for other rubber-based products decreases
by 10%. Find the level of production for each subsidiary in
order to meet this demand.

11. AN INPUT–OUTPUT MODEL FOR A THREE-SECTOR ECONOMY
Consider the economy of Exercise 1, consisting of three
sectors: agriculture (A), manufacturing (M), and trans-
portation (T), with an input–output matrix given by

$$
\begin{array}{c}
\quad A\quad\; M\quad\; T \\
\begin{array}{c}A\\M\\T\end{array}
\begin{bmatrix}
0.4 & 0.1 & 0.1 \\
0.1 & 0.4 & 0.3 \\
0.2 & 0.2 & 0.2
\end{bmatrix}
\end{array}
$$

a. Find the total output of goods needed to satisfy a con-
sumer demand for $200 million worth of agricultural
products, $100 million worth of manufactured goods,
and $60 million worth of transportation.
b. Find the value of goods and transportation consumed in
the internal process of production in order to meet this
total output.

12. AN INPUT–OUTPUT MODEL FOR A THREE-SECTOR ECONOMY
Consider a simple economy consisting of three sectors: food, clothing, and shelter. The production of 1 unit of food requires the consumption of 0.4 unit of food, 0.2 unit of clothing, and 0.2 unit of shelter. The production of 1 unit of clothing requires the consumption of 0.1 unit of food, 0.2 unit of clothing, and 0.3 unit of shelter. The production of 1 unit of shelter requires the consumption of 0.3 unit of food, 0.1 unit of clothing, and 0.1 unit of shelter. Find the level of production for each sector in order to satisfy the demand for \$100 million worth of food, \$30 million worth of clothing, and \$250 million worth of shelter.

In Exercises 13–16, matrix *A* is an input–output matrix associated with an economy, and matrix *D* (units in millions of dollars) is a demand vector. In each problem, find the final outputs of each industry such that the demands of industry and the consumer sector are met.

13. $A = \begin{bmatrix} 0.4 & 0.2 \\ 0.3 & 0.5 \end{bmatrix}$ and $D = \begin{bmatrix} 12 \\ 24 \end{bmatrix}$

14. $A = \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$

15. $A = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{5} & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 10 \\ 5 \\ 15 \end{bmatrix}$

16. $A = \begin{bmatrix} 0.2 & 0.4 & 0.1 \\ 0.3 & 0.2 & 0.1 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix}$

2.7 Solutions to Self-Check Exercises

1. Multiplying both sides of the given equation on the left by $(I - A)^{-1}$, we see that

$$X = (I - A)^{-1}D$$

Now,

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.2 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.1 \\ -0.2 & 0.8 \end{bmatrix}$$

Next, we use the Gauss–Jordan procedure to compute $(I - A)^{-1}$ (to two decimal places):

$$\begin{bmatrix} 0.6 & -0.1 & | & 1 & 0 \\ -0.2 & 0.8 & | & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{0.6}R_1}$$

$$\begin{bmatrix} 1 & -0.17 & | & 1.67 & 0 \\ -0.2 & 0.8 & | & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + 0.2R_1}$$

$$\begin{bmatrix} 1 & -0.17 & | & 1.67 & 0 \\ 0 & 0.77 & | & 0.33 & 1 \end{bmatrix} \xrightarrow{\frac{1}{0.77}R_2}$$

$$\begin{bmatrix} 1 & -0.17 & | & 1.67 & 0 \\ 0 & 1 & | & 0.43 & 1.30 \end{bmatrix} \xrightarrow{R_1 + 0.17R_2}$$

$$\begin{bmatrix} 1 & 0 & | & 1.74 & 0.22 \\ 0 & 1 & | & 0.43 & 1.30 \end{bmatrix}$$

giving

$$(I - A)^{-1} = \begin{bmatrix} 1.74 & 0.22 \\ 0.43 & 1.30 \end{bmatrix}$$

Therefore,

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = (I - A)^{-1}D = \begin{bmatrix} 1.74 & 0.22 \\ 0.43 & 1.30 \end{bmatrix} \begin{bmatrix} 50 \\ 10 \end{bmatrix} = \begin{bmatrix} 89.2 \\ 34.5 \end{bmatrix}$$

or $x = 89.2$ and $y = 34.5$.

2. a. Let

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

denote the total output matrix, where x denotes the value of the agricultural products and y the value of transportation. Also, let

$$D = \begin{bmatrix} 50 \\ 10 \end{bmatrix}$$

denote the consumer demand. Then

$$(I - A)X = D$$

or, equivalently,

$$X = (I - A)^{-1}D$$

Using the results of Exercise 1, we find that $x = 89.2$ and $y = 34.5$. That is, in order to fulfill consumer demands, \$89.2 million worth of agricultural products must be produced and \$34.5 million worth of transportation services must be used.

b. The amount of agricultural products consumed and transportation services used is given by

$$X - D = \begin{bmatrix} 89.2 \\ 34.5 \end{bmatrix} - \begin{bmatrix} 50 \\ 10 \end{bmatrix} = \begin{bmatrix} 39.2 \\ 24.5 \end{bmatrix}$$

or \$39.2 million worth of agricultural products and \$24.5 million worth of transportation services.

The Leontief Input–Output Model

Graphing Utility

Since the solution to a problem involving a Leontief input–output model often involves several matrix operations, a graphing utility can be used to facilitate the necessary computations.

APPLIED EXAMPLE 1 Suppose that the input–output matrix associated with an economy is given by A and that the matrix D is a demand vector, where

$$A = \begin{bmatrix} 0.2 & 0.4 & 0.15 \\ 0.3 & 0.1 & 0.4 \\ 0.25 & 0.4 & 0.2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 20 \\ 15 \\ 40 \end{bmatrix}$$

Find the final outputs of each industry such that the demands of industry and the consumer sector are met.

Solution First, we enter the matrices I (the identity matrix), A, and D. We are required to compute the output matrix $X = (I - A)^{-1}D$. Using the matrix operations of the graphing utility, we find (to two decimal places)

$$X = (I - A)^{-1}*D = \begin{bmatrix} 110.28 \\ 116.95 \\ 142.94 \end{bmatrix}$$

Hence the final outputs of the first, second, and third industries are 110.28, 116.95, and 142.94 units, respectively.

Excel

Here we show how to solve a problem involving a Leontief input–output model using matrix operations on a spreadsheet.

APPLIED EXAMPLE 2 Suppose that the input–output matrix associated with an economy is given by matrix A and that the matrix D is a demand vector, where

$$A = \begin{bmatrix} 0.2 & 0.4 & 0.15 \\ 0.3 & 0.1 & 0.4 \\ 0.25 & 0.4 & 0.2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 20 \\ 15 \\ 40 \end{bmatrix}$$

Find the final outputs of each industry such that the demands of industry and the consumer sector are met.

Solution

1. *Enter the elements of the matrix A and D onto a spreadsheet* (Figure T1).

	A	B	C	D	E
1		Matrix A			Matrix D
2	0.2	0.4	0.15		20
3	0.3	0.1	0.4		15
4	0.25	0.4	0.2		40

FIGURE **T1**
Spreadsheet showing matrix A and matrix D

(continued)

2. *Find* $(I - A)^{-1}$. Enter the elements of the 3×3 identity matrix I onto a spreadsheet. Highlight the cells that will contain the matrix $(I - A)^{-1}$. Type =MINVERSE (, highlight the cells containing the matrix I; type -, highlight the cells containing the matrix A; type), and press **Ctrl-Shift-Enter** . These results are shown in Figure T2.

	A	B	C
6		Matrix I	
7	1	0	0
8	0	1	0
9	0	0	1
10			
11		Matrix $(I - A)^{-1}$	
12	2.151777	1.460134	1.133525
13	1.306436	2.315082	1.402498
14	1.325648	1.613833	2.305476

FIGURE **T2**
Matrix I and matrix $(I - A)^{-1}$

3. *Compute* $(I - A)^{-1} * D$. Highlight the cells that will contain the matrix $(I - A)^{-1} * D$. Type =MMULT (, highlight the cells containing the matrix $(I - A)^{-1}$, type , , highlight the cells containing the matrix D, type), and press **Ctrl-Shift-Enter** . The resulting matrix is shown in Figure T3. So, the final outputs of the first, second, and third industries are 110.28, 116.95, and 142.94, respectively.

	A
16	Matrix $(I - A)^{-1} *D$
17	110.2786
18	116.9549
19	142.9395

FIGURE **T3**
Matrix $(I - A)^{-1}*D$

TECHNOLOGY EXERCISES

In Exercises 1–4, A is an input–output matrix associated with an economy and D (in units of million dollars) is a demand vector. Find the final outputs of each industry such that the demands of industry and the consumer sector are met.

1. $A = \begin{bmatrix} 0.3 & 0.2 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.2 & 0.3 \\ 0.3 & 0.1 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.1 & 0.2 \end{bmatrix}$ and $D = \begin{bmatrix} 40 \\ 60 \\ 70 \\ 20 \end{bmatrix}$

2. $A = \begin{bmatrix} 0.12 & 0.31 & 0.40 & 0.05 \\ 0.31 & 0.22 & 0.12 & 0.20 \\ 0.18 & 0.32 & 0.05 & 0.15 \\ 0.32 & 0.14 & 0.22 & 0.05 \end{bmatrix}$ and $D = \begin{bmatrix} 50 \\ 20 \\ 40 \\ 60 \end{bmatrix}$

3. $A = \begin{bmatrix} 0.2 & 0.2 & 0.3 & 0.05 \\ 0.1 & 0.1 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.05 & 0.2 & 0.1 \end{bmatrix}$ and $D = \begin{bmatrix} 25 \\ 30 \\ 50 \\ 40 \end{bmatrix}$

4. $A = \begin{bmatrix} 0.2 & 0.4 & 0.3 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.4 & 0.05 \\ 0.3 & 0.1 & 0.2 & 0.05 \end{bmatrix}$ and $D = \begin{bmatrix} 40 \\ 20 \\ 30 \\ 60 \end{bmatrix}$

Note: Boldfaced words/characters in a box (for example, **Enter**) indicate that an action (click, select, or press) is required. Words/characters printed blue (for example, Chart sub-type:) indicate words/characters on the screen. Words/characters printed in a typewriter font (for example, =(−2/3)*A2+2) indicate words/characters that need to be typed and entered.

CHAPTER 2   Summary of Principal Formulas and Terms

FORMULAS

1. Laws for matrix addition	
a. Commutative law	$A + B = B + A$
b. Associative law	$(A + B) + C = A + (B + C)$
2. Laws for matrix multiplication	
a. Associative law	$(AB)C = A(BC)$
b. Distributive law	$A(B + C) = AB + AC$
3. Inverse of a 2×2 matrix	If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $D = ad - bc \neq 0$ then $A^{-1} = \dfrac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
4. Solution of system $AX = B$ (A nonsingular)	$X = A^{-1}B$

TERMS

system of linear equations (68)

solution of a system of linear equations (68)

parameter (69)

dependent system (70)

inconsistent system (70)

Gauss–Jordan elimination method (76)

equivalent system (76)

coefficient matrix (78)

augmented matrix (78)

row-reduced form of a matrix (79)

row operations (80)

unit column (80)

pivoting (81)

matrix (102)

size of a matrix (102)

row matrix (102)

column matrix (102)

square matrix (102)

transpose of a matrix (106)

scalar (106)

scalar product (106)

matrix product (114)

identity matrix (117)

inverse of a matrix (128)

nonsingular matrix (128)

singular matrix (128)

input–output matrix (142)

total output matrix (142)

internal consumption matrix (143)

Leontief input–output model (144)

CHAPTER 2   Concept Review Questions

Fill in the blanks.

1. a. Two lines in the plane can intersect at (a) exactly _____ point, (b) infinitely _____ points, or (c) _____ point.

   b. A system of two linear equations in two variables can have (a) exactly _____ solution, (b) infinitely _____ solutions, or (c) _____ solution.

2. To find the point(s) of intersection of two lines, we solve the system of _____ describing the two lines.

3. The row operations used in the Gauss–Jordan elimination method are denoted by _____, _____, and _____. The use of each of these operations does not alter the _____ of the system of linear equations.

4. a. A system of linear equations with fewer equations than variables cannot have a/an _____ solution.

   b. A system of linear equations with at least as many equations as variables may have _____ solution, _____ _____ solutions, or a _____ solution.

5. Two matrices are equal provided they have the same _____ and their corresponding _____ are equal.

6. Two matrices may be added (subtracted) if they both have the same _____. To add or subtract two matrices, we add or subtract their _____ entries.

7. The transpose of a/an _____ matrix with elements a_{ij} is the matrix of size _____ with entries _____.

8. The scalar product of a matrix A by the scalar c is the matrix _____ obtained by multiplying each entry of A by _____.

152   **2** SYSTEMS OF LINEAR EQUATIONS AND MATRICES

9. a. For the product AB of two matrices A and B to be defined, the number of _____ of A must be equal to the number of _____ of B.

b. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the size of AB is _____.

10. a. If the products and sums are defined for the matrices A, B, and C, then the associative law states that $(AB)C =$ _____; the distributive law states that $A(B + C) =$ _____.

b. If I is an identity matrix of size n, then $IA = A$ if A is any matrix of size _____.

11. A matrix A is nonsingular if there exists a matrix A^{-1} such that _____ $=$ _____ $= I$. If A^{-1} does not exist, then A is said to be _____.

12. A system of n linear equations in n variables written in the form $AX = B$ has a unique solution given by $X =$ _____ if A has an inverse.

CHAPTER 2 **Review Exercises**

In Exercises 1–4, perform the operations if possible.

1. $\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$

2. $\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}$

3. $\begin{bmatrix} -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

In Exercises 5–8, find the values of the variables.

5. $\begin{bmatrix} 1 & x \\ y & 3 \end{bmatrix} = \begin{bmatrix} z & 2 \\ 3 & w \end{bmatrix}$

6. $\begin{bmatrix} 3 & x \\ y & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

7. $\begin{bmatrix} 3 & a+3 \\ -1 & b \\ c+1 & d \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ e+2 & 4 \\ -1 & 2 \end{bmatrix}$

8. $\begin{bmatrix} x & 3 & 1 \\ 0 & y & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & z \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 4 \\ 2 & 2 \end{bmatrix}$

In Exercises 9–16, compute the expressions if possible, given that

$$A = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 & 3 \\ -2 & -1 & -1 \\ 1 & 4 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 6 & 4 \\ 2 & 1 & 3 \end{bmatrix}$$

9. $2A + 3B$

10. $3A - 2B$

11. $2(3A)$

12. $2(3A - 4B)$

13. $A(B - C)$

14. $AB + AC$

15. $A(BC)$

16. $\dfrac{1}{2}(CA - CB)$

In Exercises 17–24, solve the system of linear equations using the Gauss–Jordan elimination method.

17. $\begin{aligned} 2x - 3y &= 5 \\ 3x + 4y &= -1 \end{aligned}$

18. $\begin{aligned} 3x + 2y &= 3 \\ 2x - 4y &= -14 \end{aligned}$

19. $\begin{aligned} x - y + 2z &= 5 \\ 3x + 2y + z &= 10 \\ 2x - 3y - 2z &= -10 \end{aligned}$

20. $\begin{aligned} 3x - 2y + 4z &= 16 \\ 2x + y - 2z &= -1 \\ x + 4y - 8z &= -18 \end{aligned}$

21. $\begin{aligned} 3x - 2y + 4z &= 11 \\ 2x - 4y + 5z &= 4 \\ x + 2y - z &= 10 \end{aligned}$

22. $\begin{aligned} x - 2y + 3z + 4w &= 17 \\ 2x + y - 2z - 3w &= -9 \\ 3x - y + 2z - 4w &= 0 \\ 4x + 2y - 3z + w &= -2 \end{aligned}$

23. $\begin{aligned} 3x - 2y + z &= 4 \\ x + 3y - 4z &= -3 \\ 2x - 3y + 5z &= 7 \\ x - 8y + 9z &= 10 \end{aligned}$

24. $\begin{aligned} 2x - 3y + z &= 10 \\ 3x + 2y - 2z &= -2 \\ x - 3y - 4z &= -7 \\ 4x + y - z &= 4 \end{aligned}$

In Exercises 25–32, find the inverse of the matrix (if it exists).

25. $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

26. $A = \begin{bmatrix} 2 & 4 \\ 1 & 6 \end{bmatrix}$

27. $A = \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix}$

28. $A = \begin{bmatrix} 2 & 4 \\ 1 & -2 \end{bmatrix}$

29. $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

30. $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$

31. $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 2 \\ 1 & 0 & -6 \end{bmatrix}$

32. $A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 2 & -4 \\ 3 & 1 & -2 \end{bmatrix}$

In Exercises 33–36, compute the value of the expressions if possible, given that

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

33. $(A^{-1}B)^{-1}$ **34.** $(ABC)^{-1}$

35. $(2A - C)^{-1}$ **36.** $(A + B)^{-1}$

In Exercises 37–40, write each system of linear equations in the form $AX = C$. Find A^{-1} and use the result to solve the system.

37. $2x + 3y = -8$
$\quad\ \ x - 2y = \ \ \ 3$

38. $\quad x - 3y = -1$
$\quad\ \ 2x + 4y = \ \ \ 8$

39. $\quad x - 2y + 4z = \ \ \ 13$
$\quad\ \ 2x + 3y - 2z = \quad 0$
$\quad\ \ x + 4y - 6z = -15$

40. $2x - 3y + 4z = \ \ 17$
$\quad\ x + 2y - 4z = -7$
$\quad\ 3x - \ \ y + 2z = \ \ 14$

41. GASOLINE SALES Gloria Newburg operates three self-service gasoline stations in different parts of town. On a certain day, station A sold 600 gal of premium, 800 gal of super, 1000 gal of regular gasoline, and 700 gal of diesel fuel; station B sold 700 gal of premium, 600 gal of super, 1200 gal of regular gasoline, and 400 gal of diesel fuel; station C sold 900 gal of premium, 700 gal of super, 1400 gal of regular gasoline, and 800 gal of diesel fuel. Assume that the price of gasoline was $3.20/gal for premium, $2.98/gal for super, and $2.80/gal for regular and that diesel fuel sold for $3.10/gal. Use matrix algebra to find the total revenue at each station.

42. COMMON STOCK TRANSACTIONS Jack Spaulding bought 10,000 shares of stock X, 20,000 shares of stock Y, and 30,000 shares of stock Z at a unit price of $20, $30, and $50 per share, respectively. Six months later, the closing prices of stocks X, Y, and Z were $22, $35, and $51 per share, respectively. Jack made no other stock transactions during the period in question. Compare the value of Jack's stock holdings at the time of purchase and 6 months later.

43. INVESTMENTS William's and Michael's stock holdings are given in the following table:

	BAC	GM	IBM	TRW
William	800	1200	400	1500
Michael	600	1400	600	2000

The prices (in dollars per share) of the stocks of BAC, GM, IBM, and TRW at the close of the stock market on a certain day are $50.26, $31.00, $103.07 and $38.67, respectively.
   a. Write a 2×4 matrix A giving the stock holdings of William and Michael.
   b. Write a 4×1 matrix B giving the closing prices of the stocks of BAC, GM, IBM, and TRW.
   c. Use matrix multiplication to find the total value of the stock holdings of William and Michael at the market close.

44. INVESTMENT PORTFOLIOS The following table gives the number of shares of certain corporations held by Olivia and Max in their stock portfolios at the beginning of the September and at the beginning of October:

September	IBM	Google	Boeing	GM
Olivia	800	500	1200	1500
Max	500	600	2000	800

October	IBM	Google	Boeing	GM
Olivia	900	600	1000	1200
Max	700	500	2100	900

   a. Write matrices A and B giving the stock portfolios of Olivia and Max at the beginning of September and at the beginning of October, respectively.
   b. Find a matrix C reflecting the change in the stock portfolios of Olivia and Max between the beginning of September and the beginning of October.

45. MACHINE SCHEDULING Desmond Jewelry wishes to produce three types of pendants: type A, type B, and type C. To manufacture a type-A pendant requires 2 min on machines I and II and 3 min on machine III. A type-B pendant requires 2 min on machine I, 3 min on machine II, and 4 min on machine III. A type-C pendant requires 3 min on machine I, 4 min on machine II, and 3 min on machine III. There are $3\frac{1}{2}$ hr available on machine I, $4\frac{1}{2}$ hr available on machine II, and 5 hr available on machine III. How many pendants of each type should Desmond make in order to use all the available time?

46. PETROLEUM PRODUCTION Wildcat Oil Company has two refineries, one located in Houston and the other in Tulsa. The Houston refinery ships 60% of its petroleum to a Chicago distributor and 40% of its petroleum to a Los Angeles distributor. The Tulsa refinery ships 30% of its petroleum to the Chicago distributor and 70% of its petroleum to the Los Angeles distributor. Assume that, over the year, the Chicago distributor received 240,000 gal of petroleum and the Los Angeles distributor received 460,000 gal of petroleum. Find the amount of petroleum produced at each of Wildcat's refineries.

47. INPUT–OUTPUT MATRICES The input–output matrix associated with an economy based on agriculture (A) and manufacturing (M) is given by

$$\begin{array}{cc} & \begin{matrix} A & \quad M \end{matrix} \\ \begin{matrix} A \\ M \end{matrix} & \begin{bmatrix} 0.2 & 0.15 \\ 0.1 & 0.15 \end{bmatrix} \end{array}$$

   a. Determine the amount of agricultural products consumed in the production of $200 million worth of manufactured goods.
   b. Determine the dollar amount of manufactured goods required to produce $300 million worth of all goods in the economy.
   c. Which sector consumes the greater amount of agricultural products in the production of 1 unit of goods in that sector? The lesser?

48. INPUT–OUTPUT MATRICES The matrix

$$A = \begin{bmatrix} 0.1 & 0.3 & 0.4 \\ 0.3 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}$$

is an input–output matrix associated with an economy and the matrix

$$D = \begin{bmatrix} 4 \\ 12 \\ 16 \end{bmatrix}$$

(units in millions of dollars) is a demand vector. Find the final outputs of each industry such that the demands of industry and the consumer sector are met.

49. INPUT–OUTPUT MATRICES The input–output matrix associated with an economy based on agriculture (A) and manufacturing (M) is given by

$$\begin{array}{cc} & \begin{array}{cc} A & M \end{array} \\ \begin{array}{c} A \\ M \end{array} & \begin{bmatrix} 0.2 & 0.15 \\ 0.1 & 0.15 \end{bmatrix} \end{array}$$

a. Find the gross output of goods needed to satisfy a consumer demand for $100 million worth of agricultural products and $80 million worth of manufactured goods.
b. Find the value of agricultural products and manufactured goods consumed in the internal process of production in order to meet this gross output.

CHAPTER 2 Before Moving On . . .

1. Solve the following system of linear equations, using the Gauss–Jordan elimination method:

$$\begin{aligned} 2x + \ y - \ z &= -1 \\ x + 3y + 2z &= \ \ 2 \\ 3x + 3y - 3z &= -5 \end{aligned}$$

2. Find the solution(s), if it exists, of the system of linear equations whose augmented matrix in reduced form follows.

a. $\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & -3 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 3 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 0 & -1 & | & 2 \\ 0 & 1 & 2 & | & 3 \end{bmatrix}$

3. Solve each system of linear equations using the Gauss–Jordan elimination method.
   a. $\begin{aligned} x + 2y &= \ \ 3 \\ 3x - \ y &= -5 \\ 4x + \ y &= -2 \end{aligned}$ **b.** $\begin{aligned} x - 2y + 4z &= 2 \\ 3x + \ y - 2z &= 1 \end{aligned}$

4. Let

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -2 \\ 1 & 1 \\ 3 & 4 \end{bmatrix}$$

Find (a) AB, (b) $(A + C^T)B$, and (c) $C^TB - AB^T$.

5. Find A^{-1} if

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$$

6. Solve the system

$$\begin{aligned} 2x \ \ \ \ \ + z &= \ \ 4 \\ 2x + y - z &= -1 \\ 3x + y - z &= \ \ 0 \end{aligned}$$

by first writing it in the matrix form $AX = B$ and then finding A^{-1}.