5.8: Antiderivatives

The derivative of what function \( F(x) \) will give the function \( f(x) = 2x \)?

Since the derivative of \( x^2 \) is \( 2x \), we can say the function \( F(x) = x^2 \) is the antiderivative of \( 2x \).

Note however, that \( \frac{d}{dx} (x^2 + c) = 2x \)

The function \( F(x) \) is an antiderivative of \( f(x) \) if \( F'(x) = f(x) \)

<table>
<thead>
<tr>
<th>Function</th>
<th>Antiderivative</th>
<th>Function</th>
<th>Antiderivative</th>
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</thead>
<tbody>
<tr>
<td>( k )</td>
<td>( kx + c )</td>
<td>( \sin x )</td>
<td>( -\cos x + c )</td>
</tr>
<tr>
<td>( x^n, n \neq -1 )</td>
<td>( \frac{x^{n+1}}{n+1} + c )</td>
<td>( \cos x )</td>
<td>( \sin x + c )</td>
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<td>( x^{-1} )</td>
<td>( \ln</td>
<td>x</td>
<td>+ c )</td>
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<td>( e^x )</td>
<td>( e^x + c )</td>
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<td>( f(x) + g(x) )</td>
<td>( F(x) + G(x) )</td>
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<td></td>
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<td>( \frac{1}{\sqrt{1-x^2}} )</td>
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<td>( \frac{1}{x^2+1} )</td>
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Example: Find the most general antiderivative for the following functions

(i) \( f(x) = x^3 - 4x^2 + 17 \)
\[
F(x) = \frac{x^4}{4} - \frac{4x^3}{3} + 17x + C
\]

(ii) \( f(x) = \sqrt[3]{x^2} - \sqrt[3]{x^3} = x^{\frac{2}{3}} - x \)
\[
F(x) = \frac{x^{\frac{5}{3}}}{\frac{5}{3}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C
\]

(iii) \( f(x) = \frac{x + x^2 - 1}{x^3} = \frac{x}{x^3} + \frac{x^2}{x^3} - \frac{1}{x^3} = x^{-2} + x^{-1} - x^{-3} \)
\[
F(x) = \frac{x^{-2+1}}{-2+1} + \ln |x| - \frac{x^{-3+1}}{-3+1} = -\frac{1}{x} + \ln |x| + \frac{1}{2x^2} + C
\]

(iv) \( f(x) = e^{x} + \frac{4}{\sqrt{1-x^2}} \)
\[
F(x) = e^{x} + 4 \arcsin(x) + C
\]
Example: Find \( f(x) \) given that

(i) \( f'(x) = 12x^2 - 24x + 1 \) and \( f(1) = -2 \)

\[
\begin{align*}
\int f(x) &= 4x^3 - 12x^2 + x + C \\
f(1) &= -2 = 4(1)^3 - 12(1)^2 + (1) + C \\
f(x) &= 4x^3 - 12x^2 + x + 5
\end{align*}
\]

(ii) \( f''(x) = 3e^x + 4\sin x \), \( f'(0) = 1 \), and \( f'(0) = 2 \)

\[
\begin{align*}
f'(x) &= 3e^x - 4\cos x + C \\
f'(0) &= 2 = 3e^0 - 4\cos 0 + C \implies C = 3 \\
f'(x) &= 3e^x - 4\cos x + 3 \\
f''(x) &= 3e^x - 4\sin x + 3x + d \\
f'(0) &= 1 = 3e^0 - 4\sin 0 + 3(0) + d \implies d = -2 \\
f''(x) &= 3e^x - 4\sin x + 3x - 2
\end{align*}
\]

Example: A particle is moving with acceleration \( a(t) = 3t + 8 \) m/s\(^2\). Find the position \( s(t) \) of the object at time \( t \) if we know \( s(0) = 1 \) and \( v(0) = -2 \)

\[
\begin{align*}
S(0) &= \text{clicker} \\
a(t) &= 3t + \theta = v'(t) = s''(t) \\
v(t) &= 3 \frac{t^2}{2} + 8t + C, \quad v(0) = -2 = C \implies v(t) = 3 \frac{t^2}{2} + 8t - 2 \\
s'(t) &= \left(3 \frac{t^3}{6} + 8 \frac{t^2}{2} - 2t + d\right) = \frac{t^3}{6} + 4t^2 - 2t + d \implies s(0) = 1 \\
s(t) &= \frac{t^3}{6} + 4t^2 - 2t + 1
\end{align*}
\]
**Example:** A stone is thrown downward from a 450m tall building at a speed of 5 m/s. Find a formula for the distance of the stone above ground.

\[ a(t) = -9.8 \text{ m/s}^2 = \frac{d}{dt} v(t) \Rightarrow v(t) = -9.8t + v_0 = -9.8t + 5 = s'(t) \]

\[ s(t) = -\frac{9.8t^2}{2} - 5t + s_0 = -4.9t^2 - 5t + 450 \text{ (as m m), time in sec} \]

**Example:** A car braked with constant deceleration of 40 ft/s². The skid marks produced were 160 ft before the car came to a stop. How fast was the car traveling when the brakes were first applied?

\[ a(t) = -40 = v'(t) \]

\[ v(t) = -40t + v_0 = s'(t) \]

\[ s(t) = -\frac{40t^2}{2} + v_0t + s_0 \]

\[ u(t_F) = 0 = -40t_F + v_0 \]

\[ v_0 = 40t_F \]

\[ 160 = -20t_F^2 + v_0t_F \]

\[ s(t_F) = 160 = -20t_F^2 + v_0t_F \]

\[ 160 = -20t_F^2 + 40t_F \cdot t_F = 20t_F^2 \]

\[ t_F^2 = \frac{160}{20} = 8 \Rightarrow t_F = \sqrt{8} \text{ sec} \]

\[ v_0 = 40\sqrt{8} \text{ ft/sec (77 mph)} \]