The Limit of a Function (Section 2.2)

The limit of \( f(x) \) as \( x \) approaches \( a \) is \( L \) if we can make the values of \( f(x) \) arbitrarily close to \( L \) as \( x \) is close to \( a \) but not equal to \( a \).

\[
\lim_{x \to a} f(x) = L
\]

What can we say about the \( f(x) \) in the graph on the left?

\( \lim_{x \to a^-} f(x) \) is the left-hand limit

\( \lim_{x \to a^+} f(x) \) is the right-hand limit

\[
\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L
\]

Let \( f \) be a function defined on both sides of \( a \), except possibly at \( a \) itself. Then \( \lim_{x \to a} f(x) = \infty \) means that the values of \( f(x) \) can be made arbitrarily large by taking \( x \) sufficiently close to \( a \).

Similarly, \( \lim_{x \to a} f(x) = -\infty \) when the values of \( f(x) \) can be made arbitrarily negatively large by taking \( x \) sufficiently close to \( a \).
EXAMPLE 2

Sketch the graph of the function

\[ g(x) = \begin{cases} 
2 - x & \text{if } x < -1 \\
2 & \text{if } -1 \leq x < 1 \\
x & \text{if } x = 1 \\
4 - x & \text{if } x > 1 
\end{cases} \]

Use the graph to find the following limits, if they exist:

a) \( \lim_{{x \to -1}} g(x) \)

b) \( \lim_{{x \to 1}} g(x) \)

EXAMPLE 3

Evaluate the function \( g(x) \) at the given values and then guess the value of \( \lim_{{x \to 2}} g(x) \).

\( g(x) = \frac{1 - x^2}{x^2 + 3x - 10} \)

\[ g(3) = \quad g(2.001) = \]
\[ g(2.1) = \quad g(2.0001) = \]
\[ g(2.01) = \quad g(2.00001) = \]

EXAMPLE 4

Evaluate the function \( g(x) \) at the given values and then guess the value of \( \lim_{{x \to 0}} g(x) \).

\( g(x) = \frac{\cos x - 1}{\sin x} \)

\[ g(-1) = \quad g(-0.1) = \]
\[ g(-0.5) = \quad g(-0.01) = \]
\[ g(-0.3) = \quad g(-0.001) = \]
EXAMPLE 5
Find the infinite limits

a) \( \lim_{x \to 0} \frac{x - 1}{x^2(x + 2)} \)

b) \( \lim_{x \to 5} \frac{6}{5 - x} \)

EXAMPLE 6
Find the vertical asymptotes of \( y = \frac{x}{x^2 - x - 2} \) and sketch the graph.

Limits of Vector Functions: \( \lim_{t \to a} r(t) = b \)

If \( r(t) = \langle f(t), g(t) \rangle \), then \( \lim_{t \to a} r(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t) \rangle \)

EXAMPLE 7
Given \( r(t) = \left\langle \frac{t^2 - 4t + 4}{t - 2}, \frac{t^2 - 4}{2t - 4} \right\rangle \), find \( \lim_{t \to 2} r(t) \) by finding

\( r(1.2) = \) \( r(2.8) = \)

\( r(1.5) = \) \( r(2.5) = \)

\( r(1.8) = \) \( r(2.2) = \)

\( r(1.9) = \) \( r(2.1) = \)

\( r(1.99) = \) \( r(2.01) = \)