Implicit Differentiation (Section 3.6)

**EXAMPLE 1**

Find \( \frac{dy}{dx} \) when

\[
\sqrt{x} + \sqrt{y} = 4 \quad \Rightarrow \quad \frac{dy}{dx} = 4 - \sqrt{x} \quad \Rightarrow \quad y = (4 - \sqrt{x})^2
\]

\[
y' = 2(4 - \sqrt{x}) \cdot \frac{-1}{2\sqrt{x}} = (4 - \sqrt{x}) \left( -\frac{1}{2\sqrt{x}} \right)
\]

\[
= \frac{-\sqrt{x} - 4}{\sqrt{x}} = \frac{-4}{\sqrt{x}}
\]

When \( y = f(x) \), then \( y \) is an explicit function of \( x \) and \( y' = \frac{dy}{dx} \) is straightforward. When this is not the case, we can use implicit differentiation which consists of differentiating both sides of the relation with respect to \( x \) and solving for \( y' \).

\[
\sqrt{x} + \sqrt{y} = 4 \quad \Rightarrow \quad \sqrt{y} = 4 - \sqrt{x}
\]

\[
d \left( \sqrt{x}y' + y \frac{1}{2\sqrt{x}} \right) = d \left( 4 \right)
\]

\[
\frac{1}{2\sqrt{x}} \frac{d}{dx} x + \frac{1}{2} \sqrt{y} \frac{d}{dx} y = 0
\]

\[
\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot y' = 0
\]

\[
\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot y' = 0
\]

\[
\frac{1}{2\sqrt{x}} \cdot y' = -\frac{1}{2\sqrt{x}}
\]

\[
\frac{1}{2\sqrt{x}} \cdot y' = -\frac{1}{2\sqrt{x}}
\]

\[
y' = -\frac{\sqrt{y}}{\sqrt{x}} = -\frac{(4 - \sqrt{x})}{\sqrt{x}} = 1 - 4\sqrt{x}
\]

\[
\Rightarrow \text{SAME RESULT}!
\]
EXAMPLE 2

Regard \( y \) as the independent variable and \( x \) as the dependent variable and find \( \frac{dx}{dy} \) for

\[
(x^2 + y^2)^2 = ax^2 y - y^2
\]

\[
\frac{d}{dy} (x^2 + y^2)^2 = \frac{d}{dy} (ax^2 y - y^2)
\]

\[
2(x^2 + y^2) \frac{d}{dy} (x^2 + y^2) = ax^2 \frac{dy}{dx} y + y \frac{dx}{dy} ax^2
\]

\[
2(x^2 + y^2)(2x \frac{dx}{dy} + 2y \frac{dy}{dx}) = ax^2 + ay^2 \frac{dx}{dy} x
\]

\[
2(x^2 + y^2) \left( 2x \frac{dx}{dy} + 2y \frac{dy}{dx} \right) = ax^2 + 2ax y x
\]

\[
\Rightarrow \frac{dx}{dy} = \frac{ax^2 - 4y (x^2 + y^2)}{4x (x^2 + y^2) - 2ax y}
\]

EXAMPLE 3

Find an equation of the tangent line to the curve at the given point.

\( x^{2/3} + y^{2/3} = 4 \) at \((-3\sqrt{3}, 1)\).

1. Differentiate both sides:

\[
\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \frac{dy}{dx} = 0
\]

\[
\Rightarrow \frac{dy}{dx} = -\frac{3}{\sqrt[3]{y}} \cdot \frac{3}{x}
\]

2. At \( x = -3\sqrt{3}, y = 1 \):

\[
\frac{dy}{dx} = -\frac{3}{\sqrt[3]{1}} \cdot \frac{3}{-3\sqrt{3}} = \frac{1}{\sqrt[3]{3}}
\]

3. The equation of the tangent line is:

\[
y - 1 = \frac{1}{\sqrt[3]{3}} (x + 3\sqrt{3})
\]

\[
y = \frac{1}{\sqrt[3]{3}} x + 4
\]

EXAMPLE 4

If \( [g(x)]^2 + 12x = x^2 g(x) \) and \( g(3) = 4 \), find \( g'(3) \).

\[
g(x) \cdot g'(x) + 12 = x^2 g'(x) + g(x) \cdot 2x
\]

\[
x = 3 \Rightarrow 2g(3)g'(3) + 12 = 3^2 g'(3) + g(3) \cdot 2 \cdot 3
\]

\[
8g(3) + 12 = 9g(3) + 24
\]

\[
g'(3) = -12
\]

Two curves are called orthogonal if at each point of intersection their tangent lines are perpendicular. Two families of curves are orthogonal trajectories of each other if every curve in one family is orthogonal to every curve in the other family.

EXAMPLE 5

Show that the given curves are orthogonal

\( x^2 - y^2 = 5 \), \( 4x^2 + 9y^2 = 72 \)

1. Find the intersections:

\[
y^2 = x^2 - 5 \quad \Rightarrow \quad y = \pm \sqrt{x^2 - 5}
\]

2. At \( x = \pm 3 \):

\[
y^2 = 9 - 5 = 4 \Rightarrow y = \pm 2
\]

3. The points of intersection are:

\( (3, 2), (3, -2) \)

4. Differentiate both sides:

\[
\frac{d}{dx} (x^2 - y^2) = \frac{d}{dx} (5) \Rightarrow 2x - 2y \frac{dy}{dx} = 0
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{x}{y}
\]

\[
\frac{d}{dx} (4x^2 + 9y^2) = \frac{d}{dx} (72) \Rightarrow 8x - 18y \frac{dy}{dx} = 0
\]

\[
\Rightarrow \frac{dy}{dx} = \frac{8x}{18y} = \frac{4x}{9y}
\]

\[
\frac{dy}{dx} |_{(3,2)} = \frac{3}{2} \quad \frac{dy}{dx} |_{(3,-2)} = -\frac{3}{2} \Rightarrow \perp
\]

\( (e + c) \)
**EXAMPLE 6**

Show that the given families of curves are orthogonal trajectories of each other. Sketch.

\[ y = ax^3, \quad x^2 + 3y^2 = b \]

To sketch:

Let \( a = 1 \), \( b = 16 \)

\[ y_1 = x^3 \]

\[ x^2 + 3y_2^2 = 16 \]

\[ y_1' = 3ax^2 \]

\[ 2x + 6yy_2' = 0 \]

\[ 6yy_2' = -2x \]

\[ y_2' = -\frac{x}{3y} \]

At intersection, same \((x, y)\)

\[ \Rightarrow y_2' = -\frac{x}{3(ax^3)} = -\frac{1}{3ax^2} \]

are \( \perp \)

Since \( m_1 \times m_2 = -1 \)