Derivatives and Shapes of Curves (Section 5.3)

Mean Value Theorem
If $f$ is continuous on a closed interval $[a,b]$ and differentiable on the interval $(a,b)$, then there exists a number $c$, where $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

EXAMPLE 1
Given $f(x) = 4 - x^2$, show $f(x)$ satisfies the MVT on $[1,2]$ and find all values of $c$ that satisfy the conclusion of the MVT.

EXAMPLE 2
Given that $1 \leq f'(x) \leq 4$ for all $x$ in the interval $[2,5]$, prove that $3 \leq f(5) - f(2) \leq 12$.

First Derivative Test
Suppose that $c$ is a critical number of a continuous function $f'$
- If $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$.
- If $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$.
- If $f'$ does not change sign at $c$, then $f$ does not have a local maximum or minimum at $c$.

EXAMPLE 3
Find all local extrema for the following functions
$$f(x) = x^4 - 6x^2 + 4 \quad f(x) = x\sqrt{x+1} \quad f(x) = xe^{2x}$$
Second Derivative Test
Suppose $f''$ is continuous near $c$.

- If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$
- If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$

EXAMPLE 4
Use the SDT to find the local extrema for the following functions

\[ f(x) = x^4 - 6x^2 + 4 \quad f(x) = x\sqrt{x} + 1 \quad f(x) = xe^{2x} \]

Checklist for Graphing a Function

A. Use $f(x)$ to

1) Determine the domain of the function and the intervals on which the function is continuous
2) Find all horizontal and vertical asymptotes
3) Find $x$ and $y$-intercepts

B. Use $f'(x)$ to

4) Find the critical values
5) Find intervals where the function is increasing and decreasing
6) Find all relative extrema

C. Use $f''(x)$ to

7) Find intervals where the function is concave up and concave down
8) Find all inflection points.
EXAMPLE 5
Sketch the following functions

\[ f(x) = x^4 - 6x^2 + 4 \]