The Fundamental Theorem of Calculus (Section 6.4)

At the end of chapter 5 we introduced the concept of the antiderivative:

**Antiderivatives**
The function $F(x)$ an antiderivative of $f(x)$ if $F'(x) = f(x)$

**Indefinite Integral**
If $F(x)$ is an antiderivative of $f(x)$ then $\int f(x) \, dx = F(x) + C$
We call $\int f(x) \, dx$ an indefinite integral.

**EXAMPLE 1**
Find the following indefinite integrals
(i) $\int (x^3 + 4x^2 + 8x - e) \, dx$

(ii) $\int \frac{x^2 + x - 1}{x^3} \, dx$

(iii) $\int \left[ \frac{1}{x^2 + 1} - e^x + 2^x \right] \, dx$

In the previous section we introduced the definite integral,

**Definite Integral**
If $f$ is a function defined on a closed interval $[a, b]$, let $P$ be a partition of $[a, b]$ with partition points $x_0, x_1, \ldots, x_n$ where $a = x_0 < x_1 < \ldots < x_n = b$
Choose points $x_i^*$ in the subinterval $[x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$ and $||P|| = \max \{ \Delta x_i \}$. Then the definite integral of $f$ from $a$ to $b$ is

$$\int_a^b f(x) \, dx = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$
if this limit exists. If the limit exists for $f$, then $f$ is integrable on the interval $[a, b]$.

If $f$ is positive on $[a, b]$, then the definite integral is the area under $f$ and above the $x$-axis from $x=a$ to $x=b$. If $f$ is not always positive, the definite integral is the net area.
EXAMPLE 2
Evaluate each integral by interpreting it in terms of area.

(i) \( \int_{1}^{3} c \, dt \)

(ii) \( \int_{a}^{x} c \, dt \)

(iii) \( \int_{0}^{x} t \, dt \)

THE FUNDAMENTAL THEOREM OF CALCULUS, PART 1
If \( f \) is continuous on \([a, b]\), then the function \( g \) defined by

\[
g(x) = \int_{a}^{x} f(t) \, dt \quad a \leq x \leq b
\]

is continuous on \([a, b]\), differentiable on \((a, b)\) and \( g'(x) = f(x) \)

EXAMPLE 3
Find the derivative of the given functions

(i) \( g(x) = \int_{-1}^{x} \sqrt{t^3 + 1} \, dt \)

(ii) \( g(x) = \int_{1}^{\sqrt{x}} \frac{t^2}{t^2 + 1} \, dt \)

(iii) \( g(x) = \int_{x^2}^{\pi} \frac{\sin t}{t} \, dt \)
THE FUNDAMENTAL THEOREM OF CALCULUS, PART 2
If $f$ is continuous on $[a, b]$, then
\[ \int_a^b f(x) \, dx = F(b) - F(a) = F(x)|_a^b \]
where $F$ is any antiderivative of $f$.

EXAMPLE 4
Evaluate the following definite integrals

(i) $\int_0^2 (w^3 - 1)^2 \, dw$

(ii) $\int_{-2}^0 |x^2 - 1| \, dx$

EXAMPLE 5
The velocity function (in meters per second) is given for a particle moving along a line. Find the displacement and distance traveled by the particle during the given time interval.

$v(t) = t^2 - 2t - 8, \quad 1 \leq t \leq 6$