Applied Finite Mathematics

Tomastik/Epstein

Applied Finite Mathematics, Second Edition

Edmond C. Tomastik, University of Connecticut
Janice L. Epstein, Texas A&M University

Editor: Carolyn Crockett
Editorial Assistant: Rebecca Dashiell
Marketing Manager: Myriah Fitzgibbon
Technical Editor: Mary Kanable
Illustrator: Jennifer Tribble
Photographs: Janice Epstein

Library of Congress Control Number: 2008927157
ISBN-13: 978-0-495-55533-9
ISBN-10: 0-495-55533-9

Brooks/Cole
10 Davis Drive
Belmont, CA 94002-3098
USA

Cengage Learning is a leading provider of customized learning solutions with office locations around the globe, including Singapore, the United Kingdom, Australia, Mexico, Brazil, and Japan. Locate your local office at: **international.cengage.com/region**.

Cengage Learning products are represented in Canada by Nelson Education, Ltd.

For your course and learning solutions, visit **academic.cengage.com**. Purchase any of our products at your local college bookstore or at our preferred online store **www.ichapters.com**.

Printed in the United States of America
1 2 3 4 5 6 7 12 11 10 09 08

Contents

Preface

Applied Finite Mathematics is designed for a finite mathematics course aimed at students majoring in business, management, economics, or the life or social sciences. The text can be understood by the average student with one year of high school algebra. A wide range of topics is included, giving the instructor considerable flexibility in designing a course. Optional technology material is available where relevant.

Applications truly play a central and prominent role in the text. This is because the text is written for *users* of mathematics. Thus, for example, a concrete applied problem is presented first as a motivation before developing a needed mathematical topic. After the mathematical topic has been developed, further applications are given so that the student understands the practical need for the mathematics. This is done so consistently and thoroughly that after going completing some chapters, the student should come to believe that mathematics is everywhere. Indeed, countless applications are drawn from actual referenced examples extracted from journals and other professional texts and papers.

No other skill is more important than the ability to translate a real-life problem into an appropriate mathematical format for finding the solution. Students often refer to this process as "word problems." Whereas linear systems of equations, linear programming problems, and financial problems, for example, can easily be solved using modern technology, no calculator or computer, now or in the foreseeable future, can translate these applied problems into the necessary mathematical language. Thus students, in their jobs, will most likely use their mathematical knowledge to translate applied problems into necessary mathematical models for solution by computers.

To develop these needed skills many word problems, requiring the writing of one linear equation, are given in the introductory sections. This prepares the student for the many word problems that require creating systems of linear equations. The word problems continue in subsequent chapters, for example, on linear programming.

✧ Important Features

The text can be understood by the average student with a minimum of outside assistance. Material on a variety of topics is presented in an interesting, informal, and student-friendly manner without compromising the mathematical content and accuracy. Concepts are developed gradually, always introduced intuitively, and culminate in a definition or result. Where possible, general concepts are presented only after particular cases have been presented.

Historical Boxes Scattered throughout the text, and set-off in boxes, are historical and anecdotal comments. The historical comments are not only interesting in themselves, but also indicate that mathematics is a continually developing subject.

Connections The Connection boxes relate the material to contemporary problems. This makes the material more relevant and interesting.

Applications The text includes many meaningful applications drawn from a variety of fields. For example, every section opens by posing an interesting and relevant applied problem using familiar vocabulary, which is then solved later in the section after the appropriate mathematics has been developed. Applications are given for all the mathematics that are presented and are used to motivate the student.

Worked Examples About 300 worked examples, including about 100 self-help exercises mentioned below, have been carefully selected to take the reader progressively from the simplest idea to the most complex. All the steps needed for the complete solutions are included.

Self-Help Exercises Immediately preceding each exercise set is a set of Self-Help Exercises. These approximately 100 exercises have been very carefully selected to bridge the gap between the exposition in the chapter and the regular exercise set. By doing these exercises and checking the complete solutions provided, students will be able to test or check their comprehension of the material. This, in turn, will better prepare them to do the exercises in the regular exercise set.

Exercises The text contains over 2000 exercises. Each set begins with drilling problems to build skills, and then gradually increases in difficulty. The exercise sets also include an extensive array of realistic applications from diverse disciplines. Technology exercises are included.

End of Chapter Projects Most chapters contain an in-depth exportation of an important concept taught in the chapter. This provides strong connections to real applications or a treatment of the material at a greater depth than in the main part of the chapter.

Flexibility and Technology The text does not require any technology. However, important material on how to use technology is included. This material is tucked out of the way of a reader not interested in using technology, being placed at the end of a section as technology notes and also within green boxes in the margin.

✧ Technology

For those finite math classes that are taught with a graphing calculator or a spreadsheet, this text has abundant resources for the student and the instructor. The most accessible resource is the green margin boxes with the **Technology Option**. These are designed for students who are familiar with a graphing calculator and wish to see how the current example is worked using the calculator. For those students who need step-by-step directions, the **Technology Corner** provides details on using a graphing calculator or a spreadsheet to carry out the mathematical operations discussed in the section. While the text focuses on the use of a TI-83/84 and Microsoft Excel, other technology help is available upon request.

✧ Student Aids

- **Boldface cyan text** is used when new terms are defined.
- **Boxes** are used to highlight definitions, theorems, results, and procedures.
- **Remarks** are used to draw attention to important points that might otherwise be overlooked.
- **Titles** for worked examples help to identify the subject.
- **Chapter summary outlines** at the end of each chapter conveniently summarize all the definitions, theorems, and procedures in one place.
- **Review exercises** are found at the end of each chapter.
- **Answers** to odd-numbered exercises and to all the review exercises are provided at the end of each chapter.
- A student's **solution manual** that contains completely worked solutions to all odd-numbered exercises and to all chapter review exercises is available.

✧ Instructor Aids

- An **instructor's manual** with completely worked solutions to all the exercises is available free to adopters.
- **WebAssign** A selection of questions from every section of the text will be available for online homework on the WebAssign system. These homework questions are algorithmically generated and computer graded.

✧ Content Overview

Chapter L. This chapter covers the basic topics in logic which provides a good preparation for the use of "and" and "or" in the probability applications of later chapters.
Chapter 1. The first two sections give an introduction to sets and counting the number of elements in a set. The third section then sets the background for probability by considering sample spaces and events. The next two sections then introduce the basics of probability and their rules. The next two sections cover conditional probability and Bayes' theorem.
Chapter 2. This chapter involves counting and probability. The first four sections cover the multiplication principle, permutations, combinations, probability applications of counting principles, and Bernoulli trials. The last (optional) section considers the binomial theorem.
Chapter 3. The first section revisits probability distributions and introduces histograms. The next two sections look at the measure of central tendency and the measure of the spread of data. The next sections consider the normal distribution, the approximation of the binomial distribution by a normal distribution, and finally the Poisson distribution.

Chapter 4. An introduction to the theory of the firm with some necessary economics background is provided to take into account the students' diverse backgrounds. The next three sections cover linear systems of equations. The last (optional) section on least squares provides other examples and applications of the use of linear equations.

Chapter 5. The first three sections cover the basic material on matrices. Although many applications are included in the first three sections, the fourth (optional) section is entirely devoted to input–output analysis, which is an application of linear systems and matrices used in economics.

Chapter M. The basic material on Markov processes, covering both regular and absorbing Markov processes, is presented in this chapter.

Chapter G. Game theory and its important connection to linear programming is presented in this chapter. This material gives the basics on the extensive interrelationship between linear programming and the celebrated theory of games developed by von Neumann and important in economic theory.

Chapter F. This chapter covers finance. The first two sections cover simple and compound interest. The next two sections cover annuities, sinking funds, present value, and amortization. This chapter on finance does not depend on any of the other material and can be covered at any point in the course.

✧ Some Additional Comments on the Contents

In Chapter 4 when solving systems of linear equations with an infinite number of solutions we will have free variables as parameters. We make it clear that in a list of variables, such as x, y, z, and u, the last variable need not be the free one. Rather, any of the variables can be a free variable. This requires us to develop a solution plan that can address this issue.

Also when solving a system of linear equations with an infinite number of solutions in an applied application, the parameter may require some constraints. For example, the parameter may need to be an integer, or an even integer, or have a bound above and below. Furthermore, it is possible in an applied problem that there is no acceptable solution, even though there are an infinite number of solutions of the abstract **mathematical** system.

Suppose there are three equations and three unknowns, say x, y, and z, in a system of linear equations. When using the augmented matrix to solve the system, the normal procedure is to first reduce this matrix to a matrix with ones down the diagonal and zeros below the diagonal. Students invariably notice that we now have found the z value, so why not substitute this into the previous equation, solve for y, and then use these two values to substitute into the first equation in order to find x. This is formally called backward substitution. Since this is such a natural way of solving the system, we follow backward substitution in this text. In fact, software used to solve systems follow just this plan. (See "Matrix Computations" by Gene H. Golub and Charles F. van Loan.) It does not require any more calculations than some other methods that are sometimes taught.

We also indicate in an optional subsection that following the solving plan for systems of linear equations given in this text is actually **more** efficient and in general requires **fewer** calculations than any other solving plan found in some other texts.

Acknowledgments

At the University of Connecticut we are thankful for the support offered by Michael Neumann, Jeff Tollefson, Gerald Leibowitz, and David Gross. At Texas A&M University we are thankful for the support of G. Donald Allen and the feedback of Kathryn Bollinger, Kendra Kilmer, and Heather Ramsey were invaluable. We wish to express our sincere appreciation to each of the following reviewers for their many helpful suggestions.

Marti Mclard, University of Tennessee, Knoxville
John Herron, University of Montevallo
Fritz Keineut, Iowa State University

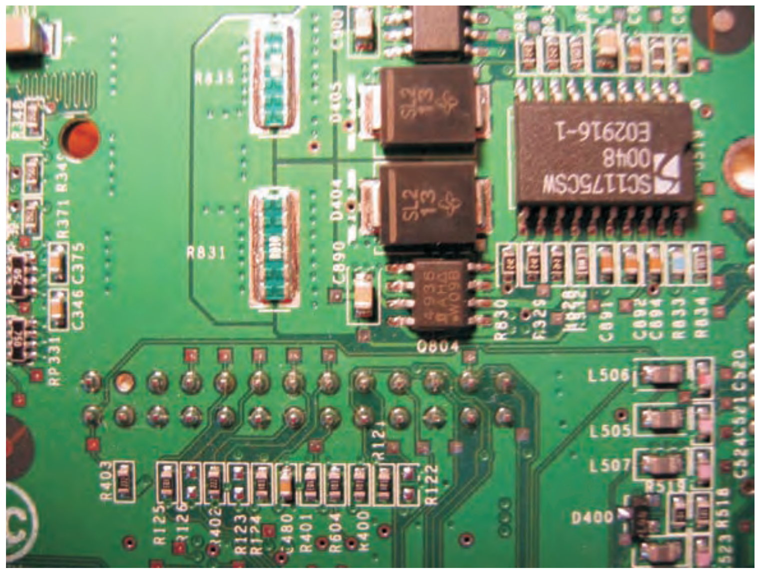
On a personal level, we both are grateful to our families for their patience and support.

Edmond C. Tomastik, University of Connecticut
Janice L. Epstein, Texas A&M University
April, 2008

Applied Finite Mathematics

CHAPTER **L**

Logic



CONNECTION

Circuit Boards

How should the circuits on this board be laid out so that the video card works? Logic is used in the design of circuit boards.

L.1 Introduction to Logic

✧ Statements

Logic is the science of correct reasoning and of making valid conclusions. In logic conclusions must be inescapable. Every concept must be clearly defined. Thus, dictionary definitions are often not sufficient since there can be no ambiguities or vagueness.

We restrict our study to declarative sentences that are unambiguous and that can be classified as true or false but not both. Such declarative sentences are called **statements** and form the basis of logic.

> **Statements**
> A **statement** is a declarative sentence that is either true or false but not both.

HISTORICAL NOTE

A Brief History of Logic

The Greek philosopher Aristotle (384–322 B.C.) is generally given the credit for the first systemic study of logic. His work, however, used ordinary language. The second great period of logic came with Gottfried Leibnitz (1646–1716), who initiated the use of symbols to simplify complicated logical arguments. This treatment is referred to as **symbolic logic** or **mathematical logic**. In symbolic logic, symbols and prescribed rules are used very much as in ordinary algebra. This frees the subject from the ambiguities of ordinary language and permits the subject to proceed and develop in a methodical way. It was, however, Augustus De Morgan (1806–1871) and George Boole (1815–1864) who systemically developed symbolic logic. The "algebra" of logic that they developed removed logic from philosophy and attached it to mathematics.

Thus, commands, questions, exclamations, and ambiguous sentences cannot be statements.

EXAMPLE 1 **Determining if Sentences Are Statements** Decide which of the following sentences are statements and which are not.

a. Look at me.
b. Do you enjoy music?
c. What a beautiful sunset!
d. Two plus two equals four.
e. Two plus two equals five.
f. The author got out of bed after 6:00 A.M. today.
g. That was a great game.
h. $x+2=5$.

Solution The first three sentences are not statements since the first is a command, the second is a question, and the third is an exclamation. Sentences **d** and **e** are statements; **d** is a true statement while **e** is a false statement. Sentence **f** is a statement, but you do not know if it is true or not. Sentence **g** is not a statement since we are not told what "great" means. With a definition of great, such as "Our team won," then it would be a statement. The last sentence **h.** is not a statement since it cannot be classified as true or false. For example, if $x=3$ it is true. But if $x=2$ it is false. ✦

✧ Connectives

A statement such as "I have money in my pocket" is called a **simple** statement since it expresses a single thought. But we need to also deal with **compound** statements such as "I have money in my pocket and my gas tank is full." We will let letters such as p, q, and r denote simple statements. To write compound statements, we need to introduce symbols for the **connectives**.

Connectives

A **connective** is a word or words, such as "and" or "if and only if," that is used to combine two or more simple statements into a compound statement.

We will consider the 3 connectives given in the following table.

Name	Connective	Symbol
Conjunction	and	$\wedge$
Disjunction	or	$\vee$
Negation	not	$\sim$

Logic does not concern itself with whether a simple statement is true or false. But if all the simple statements that make up a compound statement are known to be true or false, then the rules of logic will enable us to determine if the compound statement is true or false. We will do this in the next section.

We now carefully give the definitions of the three connectives "and," "or," and "not." Notice that the precise meanings of the three compound statements that involve these connectives are incomplete unless a clear statement is made as to when the compound statement is true and when it is false. The first connective we discuss is **conjunction** which is the concept of "and."

Conjunction

A **conjunction** is a statement of the form "p and q" and is written symbolically as

$$p \wedge q$$

The conjunction $p \wedge q$ is true if both p and q are true, otherwise it is false.

EXAMPLE 2 **Using Conjunction** Write the compound statement "I have money in my pocket and my gas tank is full" in symbolic form.

Solution First let p be the statement "I have money in my pocket" and q be the statement "my gas tank is full." Since $\wedge$ represents the word "and," the compound statement can be written symbolically as $p \wedge q$. ✦

The next connective we consider is **disjunction** which is the concept of "or." Make careful note of the fact that the logical "or" is slightly different in meaning than the typical English use of the word "or."

Disjunction

A **disjunction** is a statement of the form "p or q" and is written symbolically as

$$p \vee q$$

The disjunction $p \vee q$ is false if both p and q are false and is true in all other cases.

REMARK: The word "or" in this definition conveys the meaning "one or the other, or both." This is also called the **inclusive or**.

EXAMPLE 3 **Using Disjunction** Write the compound statement "Janet is in the top 10% of her class or she lives on campus" in symbolic form.

Solution First let p be the statement "Janet is in the top 10% of her class" and q the statement "She lives on campus." Since $\vee$ represents the word "or," the compound statement can be written as $p \vee q$. ✦

REMARK: In everyday language the word "or" is not always used in the way indicated above. For example, if a car salesman tells you that for $20,000 you can have a new car with automatic transmission or a new car with air conditioning, he means "one or the other, but not both." This use of the word "or" is called **exclusive or**.

The final connective introduced in this section is **negation** which is the concept of "not."

Negation

Negation
A **negation** is a statement of the form "not p" and is written symbolically as

$$\sim p$$

The negation $\sim p$ is true if p is false and false if p is true.

For example, if p is the statement "Janet is smart," then $\sim p$ is the statement "Janet is not smart."

EXAMPLE 4 **Using Negation** Let p and q be the following statements:

p: George Bush plays football for the Washington Redskins.
q: The Dow Jones industrial average set a new record high last week.

Write the following statements in symbolic form.

a. George Bush does not play football for the Washington Redskins, and the Dow Jones industrial average set a new record high last week.

b. George Bush plays football for the Washington Redskins, or the Dow Jones industrial average did not set a new record high last week.

c. George Bush does not play football for the Washington Redskins, and the Dow Jones industrial average did not set a new record high last week.

d. It is not true that George Bush plays football for the Washington Redskins and that the Dow Jones industrial average set a new record high last week.

Solution **a.** $(\sim p) \wedge q$ **b.** $p \vee \sim q$ **c.** $\sim p \wedge \sim q$ **d.** $\sim (p \wedge q)$ ✦

EXAMPLE 5 **Translating Symbolic Forms Into Compound Statements** Let p and q be the following statements:

p: Philadelphia is the capital of New Jersey.
q: General Electric lost money last year.

Write out the statements that correspond to each of the following:

a. $p \vee q$ **b.** $p \wedge q$ **c.** $p \vee \sim q$ **d.** $\sim p \wedge \sim q$

Solution

a. Philadelphia is the capital of New Jersey, or General Electric lost money last year.

b. Philadelphia is the capital of New Jersey, and General Electric lost money last year.

c. Philadelphia is the capital of New Jersey, or General Electric did not lose money last year.

d. Philadelphia is not the capital of New Jersey, and General Electric did not lose money last year.

✦

In most cases when dealing with complex compound statements, there will not be a question as to the order in which to apply the connectives. However, you may have noticed in the above examples that negation was used before disjunction or conjunction. The order of precedence for logical connectives is stated below.

Order of Precedence

The logical connectives are used in the following order

$$\sim, \wedge, \vee$$

Self-Help Exercises L.1

1. Determine which of the following sentences are statements:

a. The Atlanta Braves won the World Series in 1992.

b. IBM makes oil tankers for Denmark.

c. Does IBM make oil tankers for Denmark?

d. Please pay attention.

e. I have a three-dollar bill in my purse, or I don't have a purse.

2. Let p be the statement "George Washington was never president of the United States" and q be the statement "George Washington wore a wig." Write out the statements that correspond to the following:

a. $\sim p$ **b.** $p \vee q$ **c.** $\sim p \wedge q$
d. $p \wedge \sim q$ **e.** $\sim p \vee \sim q$

L.1 Exercises

In Exercises 1 through 14, decide which are statements.

1. Water freezes at 70^{o}F.

2. It rained in St. Louis on May 4, 1992.

3. $5 > 10$.

4. This sentence is false.

5. The number 4 is not a prime.

6. How are you feeling?

7. I feel great!

8. $10 + 10 - 5 = 25$

9. There is life on Mars.

10. Cleveland is the largest city in Ohio.

11. Who said Cleveland is the largest city in Ohio?

12. You don't say!

13. IBM lost money in 1947.

14. Groundhog Day is on February 12.

15. Let p and q denote the following statements:

p: George Washington was the third president of the United States.

q: Austin is the capital of Texas.

Express the following compound statements in words:

a. $\sim p$ **b.** $p \wedge q$ **c.** $p \vee q$

d. $\sim p \wedge q$ **e.** $p \vee \sim q$ **f.** $\sim (p \wedge q)$

16. Let p and q denote the following statements:

p: Mount McKinley is the highest point in the United States.

q: George Washington was a signer of the Declaration of Independence.

Express the following compound statements in words.

a. $\sim q$ **b.** $p \wedge q$ **c.** $p \vee q$

d. $p \wedge \sim q$ **e.** $\sim p \wedge \sim q$ **f.** $\sim (p \vee q)$

17. Let p and q denote the following statements:

p: George Washington owned over 100,000 acres of property.

q: The Exxon Valdez was a luxury liner.

a. State the negation of these statements in words.

b. State the disjunction for these statements in words.

c. State the conjunction for these statements in words.

18. Let p and q denote the following statements:

p: McDonald's Corporation operates large farms.

q: Wendy's Corporation operates fast-food restaurants.

a. State the negation of these statements in words.

b. State the disjunction for these statements in words.

c. State the conjunction for these statements in words.

19. Let p and q denote the following statements:

p: The *Wall Street Journal* has the highest daily circulation of any newspaper.

q: *Advise and Consent* was written by Irving Stone.

Give a symbolic expression for the statements below.

a. *Advise and Consent* was not written by Irving Stone.

b. The *Wall Street Journal* has the highest daily circulation of any newspaper, and *Advise and Consent* was not written by Irving Stone.

c. The *Wall Street Journal* has the highest daily circulation of any newspaper, or *Advise and Consent* was written by Irving Stone.

d. The *Wall Street Journal* does not have the highest daily circulation of any newspaper, or *Advise and Consent* was not written by Irving Stone.

20. Let p and q denote the following statements:

p: IBM makes computers.

q: IBM makes trucks.

Give a symbolic expression for the statements below.

a. IBM does not make trucks.

b. IBM makes computers, or IBM makes trucks.

c. IBM makes computers, or IBM does not make trucks.

d. IBM does not make computers, and IBM does not make trucks.

Solutions to Self-Help Exercises L.1

1. The sentences **a**, **b**, and **e** are statements, while **c** and **d** are not.

2. **a.** George Washington was a president of the United States.

b. George Washington was never president of the United States, or George Washington wore a wig.

c. George Washington was a president of the United States, and George Washington wore a wig.

d. George Washington was never president of the United States, and George Washington did not wear a wig.

e. George Washington was a president of the United States, or George Washington did not wear a wig.

L.2 Truth Tables

✧ Introduction to Truth Tables

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table L.1

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table L.2

p	$\sim p$
T	F
F	T

Table L.3

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Table L.4

The **truth value** of a statement is either true or false. Thus the statement "Ronald H. Coarse won the Nobel Prize in Economics in 1991" has truth value true since it is a true statement, whereas the statement "Los Angeles is the capital of California" has truth value false since it is a false statement.

Logic does not concern itself with the truth value of simple statements. But if we know the truth values of the simple statements that make up a compound statement, then logic can determine the truth value of the compound statement.

For example, to understand the very definition of $p \vee q$, one must know under what conditions the compound statement will be true. As defined in the last section $p \vee q$ is always true unless both p and q are false. A convenient way of summarizing this is by a truth table. This is done in Table L.1.

The truth tables for the statements $p \wedge q$ and $\sim p$ are given in Table L.2 and Table L.3. As Table L.2 indicates, $p \wedge q$ is true only if both p and q are true. Given a general compound statement, we wish to determine the truth value given any possible combination of truth values for the simple statements that are contained in the compound statement. We use a truth table for this purpose. The next examples illustrate how this is done.

EXAMPLE 1 **Constructing a Truth Table** Construct a truth table for the statement $p \vee \sim q$.

Solution Place p and q at the head of the first two columns as indicated in Table L.4 $\vee$ found in Table L.1. ✦ and list all possible truth values for p and q as indicated. It is strongly recommended that you always list the truth values in the first two columns in the same way. This will be particularly useful later when we will need to compare two truth tables. Now enter the truth values for $\sim q$ in the third column. Now using the first and third columns of the table, construct the fourth column using the definition of

EXAMPLE 2 **Constructing a Truth Table** Construct a truth table for the statement $\sim p \wedge (p \vee q)$.

Solution Make the same first two columns as before. Next make a column for $\sim p$ and the corresponding truth values. Now make a fourth column for $p \vee q$. Finally, using the third and fourth columns and the definition of $\wedge$, fill in the fifth column of Table L.5.

p	q	$\sim p$	$p \vee q$	$\sim p \wedge (p \vee q)$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

Table L.5

Thus we see that $\sim p \wedge (p \vee q)$ is true only if p is false and q is true. ✦

We can construct a truth table for a compound statement with three simple statements.

EXAMPLE 3 **Constructing a Truth Table** Construct a truth table for the statement $(p \wedge q) \wedge [(r \vee \sim p) \wedge q]$.

Solution Always use the same order of T's and F's that are indicated in the first three columns of Table L.6. Fill in the rest of the columns in the order given.

p	q	r	$p \wedge q$	$\sim p$	$r \vee \sim p$	$(r \vee \sim p) \wedge q$	$(p \wedge q) \wedge [(r \vee \sim p) \wedge q]$
T	T	T	T	F	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	T	F	T	T	T	F
F	T	F	F	T	T	T	F
F	F	T	F	T	T	F	F
F	F	F	F	T	T	F	F

Table L.6

We see that $(p \wedge q) \wedge [(r \vee \sim p) \wedge q]$ is true only if p, q, and r are all true. ✦

✧ Exclusive Disjunction

We now consider the exclusive "or." Recall that the exclusive "or" means "one or the other, but not both." The truth table for the exclusive disjunction is given in Table L.7 where we note that the symbol for the exclusive disjunction is $\underline{\vee}$. Notice that $\underline{\vee}$ is true only if exactly one of the two statements is true.

p	q	$p \underline{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

Table L.7

REMARK: Unless clearly specified otherwise, the word "or" will always be taken in the *in*clusive sense.

EXAMPLE 4 **Determining the Truth Value of a Statement** Let p and q be the following statements:

p: Aaron Copland was an American composer.
q: Rudolf Serkin was a violinist.

Determine the truth value of each of the following statements:

a. $p \vee q$ **b.** $\sim (p \vee q)$ **c.** $p \underline{\vee} q$ **d.** $\sim (p \underline{\vee} q)$ **e.** $p \wedge \sim q$

Solution First note that p is true and q is false.[1] Both the disjunction in **a** and the exclusive disjunction in **c** are therefore true. Thus their negations in **b** and **d** are false. The statement in **e** is the conjunction of a true statement p with a true statement $\sim q$ and thus is true. ✦

✧ Tautology and Contradiction

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Table L.8

The statement $p \wedge \sim p$ is always false according to the truth table in Table L.8. In such a case, we say that the statement $p \wedge \sim p$ is a **contradiction**. If the statement is always true, we say that the statement is a **tautology**.

Contradiction and Tautology
We say that a statement is a **contradiction** if the truth value of the statement is always false no matter what the truth values of its simple component statements. We say that a statement is a **tautology** if the truth value of the statement is always true no matter what the truth values of its simple component statements.

EXAMPLE 5 Determining if a Statement Is a Tautology Determine if the statement $p \wedge (\sim p \wedge q)$ is a tautology.

Solution Create a truth table.

p	q	$\sim p$	$\sim p \vee q$	$p \wedge (\sim p \vee q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

The truth table indicates that the statement is true no matter what the truth values of p and q are. Thus, this statement is a tautology. ✦

Self-Help Exercises L.2

1. Construct the truth table for the statement

$$(p \vee q) \vee (r \wedge \sim q)$$

2. Let p be the statement "George Washington was the first president of the United States" and q be the statement "George Washington wore a wig." Determine the truth value of each of the statements below.

a. $\sim p$ **b.** $p \vee q$ **c.** $\sim p \wedge q$
d. $p \wedge \sim q$ **e.** $\sim p \vee \sim q$

[1] Serkin was a famous pianist.

L.2 Exercises

In Exercises 1 through 20, construct a truth table for the given statement. Indicate if a statement is a tautology or a contradiction.

1. $p \wedge \sim q$ **2.** $p \underline{\vee} \sim q$

3. $\sim (\sim p)$ **4.** $\sim (p \wedge q)$

5. $(p \wedge \sim q) \vee q$ **6.** $(p \underline{\vee} q) \vee \sim q$

7. $\sim p \vee (p \wedge q)$ **8.** $\sim p \underline{\vee} (p \wedge q)$

9. $(p \vee q) \wedge (p \wedge q)$ **10.** $(p \wedge q) \vee (p \vee q)$

11. $(p \vee \sim q) \vee (\sim p \wedge q)$ **12.** $(p \wedge \sim q) \wedge (p \vee \sim q)$

13. $(p \vee q) \wedge r$ **14.** $p \vee (q \wedge r)$

15. $\sim [(p \wedge q) \wedge r]$ **16.** $\sim [p \wedge (q \wedge r)]$

17. $(p \vee q) \vee (q \wedge r)$ **18.** $(p \wedge q) \wedge (q \vee r)$

19. $(p \vee \sim q) \vee (\sim q \wedge r)$ **20.** $(\sim p \wedge q) \wedge (\sim q \underline{\vee} r)$

21. Let p and q be the statements:

p: Roe v. Wade was a famous boxing match.

q: Iraq invaded Kuwait in 1990.

Determine the truth value of the following compound statements:

a. $\sim p$ **b.** $p \wedge q$ **c.** $p \vee q$

d. $\sim p \wedge q$ **e.** $p \vee \sim q$

Note that p is false and q is true.

22. Let p and q be the statements:

p: The sun rises in the east.

q: Proctor & Gamble is a casino in Las Vegas.

Determine the truth value of the following compound statements:

a. $\sim q$ **b.** $p \wedge q$ **c.** $p \underline{\vee} q$

d. $p \vee \sim q$ **e.** $\sim p \vee q$

23. Let p and q be the statements:

p: The South Pole is the southernmost point on the Earth.

q: The North Pole is a monument in Washington, D.C.

Determine the truth value of the following compound statements:

a. $\sim q$ **b.** $p \underline{\vee} \sim q$ **c.** $\sim p \wedge q$

d. $\sim (p \wedge q)$

24. Let p and q be the statements:

p: Stevie Wonder is a famous singer.

q: Simon & Garfunkel is a famous law firm.

Determine the truth value of the following compound statements:

a. $p \wedge \sim q$ **b.** $p \underline{\vee} q$ **c.** $p \vee q$

d. $\sim (p \vee q)$

Note that p is true and q is false.

Solutions to Self-Help Exercises L.2

1.

p	q	r	$p \vee q$	$\sim q$	$r \wedge \sim q$	$(p \vee q) \vee (r \wedge \sim q)$
T	T	T	T	F	F	T
T	T	F	T	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	F	F	T
F	T	F	T	F	F	T
F	F	T	F	T	T	T
F	F	F	F	T	F	F

2. The statement p is true, while q is false. Thus

a. $\sim p$ is false. **b.** $p \vee q$ is true. **c.** $\sim p \wedge q$ is false.

d. $p \wedge \sim q$ is true is $\sim q$ is true.

e. $\sim p \vee \sim q$ is true.

CHAPTER 1

Sets and Probability

In a survey of 200 people that had just returned from a trip to Europe, the following information was gathered.

- 142 visited England
- 95 visited Italy
- 65 visited Germany
- 70 visited both England and Italy
- 50 visited both England and Germany
- 30 visited both Italy and Germany
- 20 visited all three of these countries

How many went to England but not Italy or Germany?

We will learn how to solve puzzles like this in the second section of the chapter when counting the elements in a set is discussed.

1.1 Introduction to Sets

This section discusses operations on sets and the laws governing these set operations. These are fundamental notions that will be used throughout the remainder of this text. In the next two chapters we will see that probability and statistics are based on counting the elements in sets and manipulating set operations. Thus we first need to understand clearly the notion of sets and their operations.

✧ The Language of Sets

We begin here with some definitions of the language and notation used when working with sets. The most basic definition is "What is a set?" A **set** is a collection of items. These items are referred to as the **elements** or **members** of the set. For example, the set containing the numbers 1, 2, and 3 would be written $\{1,2,3\}$. Notice that the set is contained in curly brackets. This will help us distinguish sets from other mathematical objects.

When all the elements of the set are written out, we refer to this as **roster notation**. So the set containing the first 10 letters in the English alphabet would be written as $\{a,b,c,d,e,f,g,h,i,j\}$ in roster notation. If we wanted to refer to this set without writing all the elements, we could define the set in terms of its properties. This is called **set-builder notation**. So we write

$$\{x|x \text{ is one of the first 10 letters in the English alphabet}\}$$

This is read "the set of all x such that x is one of the first 10 letters in the English alphabet". If we will be using a set more than once in a discussion, it is useful to define the set with a symbol, usually an uppercase letter. So

$$S = \{a,b,c,d,e,f,g,h,i,j\}$$

We can say c is an element of the set $\{a,b,c,d,e,f,g,h,i,j\}$ or simply write $c \in S$. The symbol $\in$ is read "is an element of". We can also say that the set $R = \{c\}$ is a subset of our larger set S as every element in the set R is also in the set S.

> **Subsets**
> If every element of a set A is also an element of another set B, we say that A is a **subset** of B and write $A \subseteq B$. If A is not a subset of B, we write $A \not\subseteq B$.

Thus $\{1,2,4\} \subseteq \{1,2,3,4\}$, but $\{1,2,3,4\} \not\subseteq \{1,2,4\}$. Since every element in A is in A, we can write $A \subseteq A$. If there is a set B and every element in the set B is also in the set A but $B \neq A$, we say that B is a **proper subset** of A. This is written as $B \subset A$. Note the proper subset symbol $\subset$ is lacking the small horizontal line that the subset symbol $\subseteq$ has. The difference is rather like the difference between $<$ and $\leq$.

Some sets have no elements at all. We need some notation for this, simply leaving a blank space will not do!

> *HISTORICAL NOTE*
>
> **George Boole, 1815–1864**
>
> George Boole was born into a lower-class family in Lincoln, England, and had only a common school education. He was largely self-taught and managed to become an elementary school teacher. Up to this time any rule of algebra such as $a(x+y) = ax+ay$ was understood to apply only to numbers and magnitudes. Boole developed an "algebra" of sets where the elements of the sets could be not just numbers but anything. This then laid down the foundations for a fundamental way of thinking. Bertrand Russell, a great mathematician and philosopher of the 20th century, said that the greatest discovery of the 19th century was the nature of pure mathematics, which he asserted was discovered by George Boole. Boole's pamphlet "The Mathematical Analysis of Logic" maintained that the essential character of mathematics lies in its form rather than in its content. Thus mathematics is not merely the science of measurement and number but any study consisting of symbols and precise rules of operation. Boole founded not only a new algebra of sets but also a formal logic that we will discuss in Chapter L.

Empty Set
The **empty set**, written as $\emptyset$ or $\{\}$, is the set with no elements.

The empty set can be used to conveniently indicate that an equation has no solution. For example

$$\{x|x \text{ is real and } x^2 = -1\} = \emptyset$$

By the definition of subset, given any set A, we must have $\emptyset \subseteq A$.

EXAMPLE 1 Finding Subsets Find all the subsets of $\{a,b,c\}$.

Solution The subsets are

$$\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$$ ✦

REMARK: Note that there are 8 subsets and 7 of them are proper subsets. In general, a set with n elements will have 2^n subsets. In the next chapter we will learn why this is so.

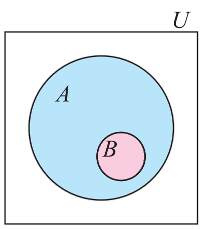


Figure 1.1

The empty set is the set with no elements. At the other extreme is the **universal set**. This set is the set of all elements being considered and is denoted by U. If, for example, we are to take a national survey of voter satisfaction with the president, the universal set is the set of all voters in this country. If the survey is to determine the effects of smoking on pregnant women, the universal set is the set of all pregnant women. The context of the problem under discussion will determine the universal set for that problem. The universal set must contain every element under discussion.

A **Venn diagram** is a way of visualizing sets. The universal set is represented by a rectangle and sets are represented as circles inside the universal set. For example, given a universal set U and a set A, Figure 1.1 is a Venn diagram that visualizes the concept that $A \subset U$. Figure 1.1 also visualizes the concept $B \subset A$. The U above the rectangle will be dropped in later diagrams as we will abide by the convention that the rectangle always represents the universal set.

✧ Set Operations

The first set operation we consider is the complement. The complement of set A are those members of set U that do **not** belong to A.

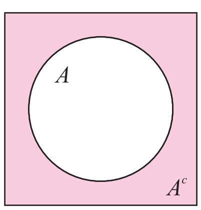


Figure 1.2
A^c is shaded.

Complement
Given a universal set U and a set $A \subset U$, the **complement of A**, written A^c, is the set of all elements that are in U but not in A, that is,

$$A^c = \{x|x \in U,\ x \notin A\}$$

A Venn diagram visualizing A^c is shown in Figure 1.2. Some alternate notations for the complement of a set are A' and $\overline{A}$.

EXAMPLE 2 **The Complements of Sets** Let $U = \{1,2,3,4,5,6,7,8,9\}$, $A = \{1,3,5,7,9\}$, $B = \{1,2,3,4,5\}$. Find A^c, B^c, U^c, $\emptyset^c$, and $(A^c)^c$ in roster notation.

Solution We have

$$\begin{aligned} A^c &= \{2,4,6,8\} \\ B^c &= \{6,7,8,9\} \\ U^c &= \emptyset \\ \emptyset^c &= \{1,2,3,4,5,6,7,8,9\} = U \\ (A^c)^c &= \{2,4,6,8\}^c \\ &= \{1,3,5,7,9\} = A \end{aligned}$$

✦

Note that in the example above we found $U^c = \emptyset$ and $\emptyset^c = U$. Additionally $(A^c)^c = A$. This can be seen using the Venn diagram in Figure 1.2, since the complement of A^c is all elements in U but not in A^c which is the set A. These three rules are called the **Complement Rules**.

Complement Rules

If U is a universal set, we must always have

$$U^c = \emptyset, \qquad \emptyset^c = U$$

If A is any subset of a universal set U, then

$$(A^c)^c = A$$

The next set operation is the union of two sets. This set includes the members of both sets A and B. That is, if an element belongs to set A or set B then it belongs to the union of A and B.

Set Union

The **union** of two sets A and B, written $A \cup B$, is the set of all elements that belong to A, or to B, or to both. Thus

$$A \cup B = \{x | x \in A \text{ or } x \in B \text{ or both}\}$$

REMARK: This usage of the word “or” is the same as in logic. It is the inclusive “or” where the elements that belong to both sets are part of the union. In English the use of “or” is often the exclusive “or”. That is, if a meal you order at a restaurant comes with a dessert and you are offered cake or pie, you really only get one of the desserts. Choosing one dessert will exclude you from the other. If it was the logical “or” you could have both!

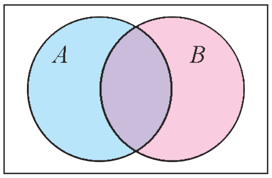


Figure 1.3
$A \cup B$ is shaded.

Our convention will be to drop the phrase "or both" but still maintain the same meaning. Note very carefully that this gives a particular definition to the word "or". Thus we will normally write

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

It can be helpful to say that the union of A and B, $A \cup B$, is all elements in A joined together with all elements in B. A Venn diagram visualizing this is shown in Figure 1.3 with the union shaded.

EXAMPLE 3 **The Union of Two Sets** Let $U = \{1,2,3,4,5,6\}$, $A = \{1,2,3,4\}$ and $B = \{1,4,5,6\}$. Find $A \cup B$ and $A \cup A^c$.

Solution We begin with the first set and join to it any elements in the second set that are not already there. Thus

$$\begin{aligned} A \cup B &= \{1,2,3,4\} \cup \{1,4,5,6\} \\ &= \{1,2,3,4,5,6\} \end{aligned}$$

Since $A^c = \{5,6\}$ we have

$$A \cup A^c = \{1,2,3,4\} \cup \{5,6\} = \{1,2,3,4,5,6\} = U$$ ✦

The second result, $A \cup A^c = U$ is generally true. From Figure 1.2, we can see that if U is a universal set and $A \subset U$, then

$$A \cup A^c = U$$

Set Intersection
The **intersection** of two sets A and B, written $A \cap B$, is the set of all elements that belong to both the set A and to the set B. Thus

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

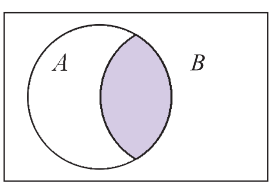


Figure 1.4

A Venn diagram is shown in Figure 1.4 with the intersection shaded.

EXAMPLE 4 **The Intersection of Two Sets** Find

a. $\{a,b,c,d\} \cap \{a,c,e\}$ **b.** $\{a,b\} \cap \{c,d\}$

Solution **a.** Only a and c are elements of both of the sets. Thus

$$\{a,b,c,d\} \cap \{a,c,e\} = \{a,c\}$$

b. The two sets $\{a,b\}$ and $\{c,d\}$ have no elements in common. Thus

$$\{a,b\} \cap \{c,d\} = \emptyset$$ ✦

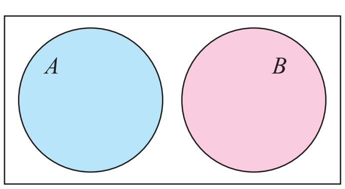


Figure 1.5
A and B are disjoint.

The sets $\{a,b\}$ and $\{c,d\}$ have no elements in common. These sets are called disjoint and can be visualized in Figure 1.5.

Disjoint Sets
Two sets A and B are **disjoint** if they have no elements in common, that is, if $A \cap B = \emptyset$.

An examination of Figure 1.2 or referring to the definition of A^c indicates that for any set A, A and A^c are disjoint. That is,

$$A \cap A^c = \emptyset$$

✧ Additional Laws for Sets

There are a number of laws for sets. They are referred to as commutative, associative, distributive, and De Morgan laws. We will consider two of these laws in the following examples.

HISTORICAL NOTE

Augustus De Morgan, 1806–1871

It was De Morgan who got George Boole interested in set theory and formal logic and then made significant advances upon Boole's epochal work. He discovered the De Morgan laws referred to in the last section. Boole and De Morgan are together considered the founders of the algebra of sets and of mathematical logic. De Morgan was a champion of religious and intellectual toleration and on several occasions resigned his professorships in protest of the abridgments of academic freedom of others.

EXAMPLE 5 **Establishing a De Morgan Law** Use a Venn diagram to show that

$$(A \cup B)^c = A^c \cap B^c$$

Solution We first consider the right side of this equation. Figure 1.6 shows a Venn diagram of A^c and B^c and $A^c \cap B^c$. We then notice from Figure 1.3 that this is $(A \cup B)^c$.

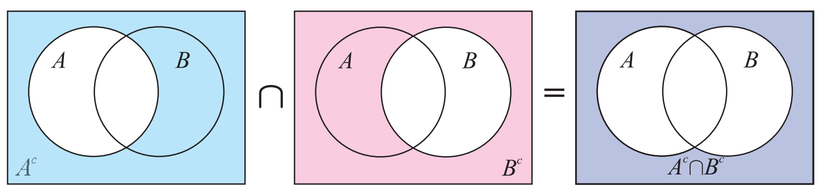


Figure 1.6 ✦

EXAMPLE 6 **Establishing the Distributive Law for Union** Use a Venn diagram to show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution Consider first the left side of this equation. In Figure 1.7a the sets A, $B \cap C$, and the union of these two are shown. Now for the right side of the equation refer to Figure 1.7b, where the sets $A \cup B$, $A \cup C$, and the intersection of these two sets are shown. We have the same set in both cases. ✦

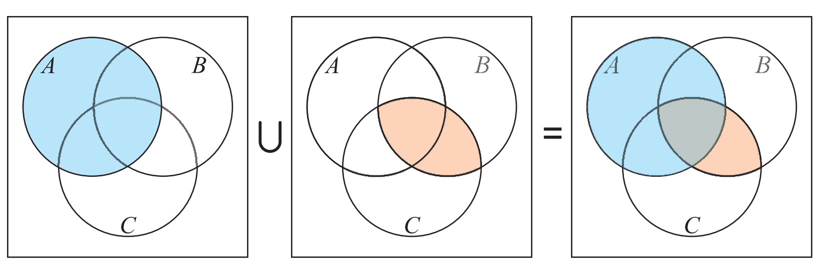


Figure 1.7a

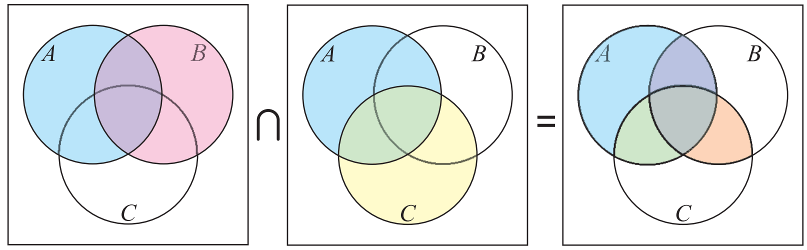


Figure 1.7b

We can summarize the laws we have found in the following list.

Laws for Set Operations

$A \cup B = B \cup A$	Commutative law for union
$A \cap B = B \cap A$	Commutative law for intersection
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative law for union
$A \cap (B \cap C) = (A \cap B) \cap C$	Associative law for intersection
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive law for union
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive law for intersection
$(A \cup B)^c = A^c \cap B^c$	De Morgan law
$(A \cap B)^c = A^c \cup B^c$	De Morgan law

✧ Applications

EXAMPLE 7 Using Set Operations to Write Expressions Let U be the universal set consisting of the set of all students taking classes at the University of Hawaii and

$B = \{x | x \text{ is currently taking a business course}\}$
$E = \{x | x \text{ is currently taking an English course}\}$
$M = \{x | x \text{ is currently taking a math course}\}$

Write an expression using set operations and show the region on a Venn diagram for each of the following:

a. The set of students at the University of Hawaii taking a course in at least one of the above three fields.

b. The set of all students at the University of Hawaii taking both an English course and a math course but not a business course.

c. The set of all students at the University of Hawaii taking a course in exactly one of the three fields above.

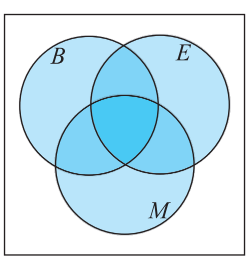


Figure 1.8a

Solution

a. This is $B \cup E \cup M$. See Figure 1.8a.

b. This can be described as the set of students taking an English course (E) and also (intersection) a math course (M) and also (intersection) not a business course (B^c) or

$$E \cap M \cap B^c$$

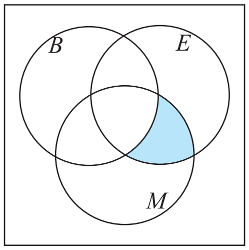


Figure 1.8b

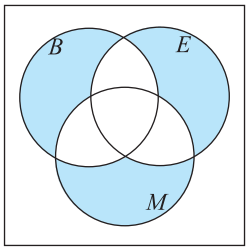


Figure 1.8c

This is the set of points in the universal set that are in both E and M but not in B and is shown in Figure 1.8b.

c. We describe this set as the set of students taking business but not taking English or math ($B \cap E^c \cap M^c$) together with (union) the set of students taking English but not business or math ($E \cap B^c \cap M^c$) together with (union) the set of students taking math but not business or English ($M \cap B^c \cap E^c$) or

$$(B \cap E^c \cap M^c) \cup (B^c \cap E \cap M^c) \cup (B^c \cap E^c \cap M)$$

This is the union of the three sets shown in Figure 1.8c. The first, $B \cap E^c \cap M^c$, consists of those points in B that are outside E and also outside M. The second set $E \cap B^c \cap M^c$ consists of those points in E that are outside B and M. The third set $M \cap B^c \cap E^c$ is the set of points in M that are outside B and E. The union of these three sets is then shown on the right in Figure 1.8c. ✦

REMARK: The word **only** means the same as exactly one. So a student taking only a business course would be written as $B \cap E^c \cap M^c$.

Self-Help Exercises 1.1

1. Let $U = \{1,2,3,4,5,6,7\}$, $A = \{l,2,3,4\}$, $B = \{3,4,5\}$, $C = \{2,3,4,5,6\}$. Find the following:

a. $A \cup B$ **b.** $A \cap B$
c. A^c **d.** $(A \cup B) \cap C$
e. $(A \cap B) \cup C$ **f.** $A^c \cup B \cup C$

2. Let U denote the set of all corporations in this country and P those that made profits during the last year, D those that paid a dividend during the last year, and L those that increased their labor force during the last year. Describe the following using the three sets P, D, L, and set operations. Show the regions in a Venn diagram.

a. Corporations in this country that had profits and also paid a dividend last year
b. Corporations in this country that either had profits or paid a dividend last year
c. Corporations in this country that did not have profits last year
d. Corporations in this country that had profits, paid a dividend, and did not increase their labor force last year
e. Corporations in this country that had profits or paid a dividend, and did not increase their labor force last year

1.1 Exercises

In Exercises 1 through 4, determine whether the statements are true or false.

1. a. $\emptyset \in A$ **b.** $A \in A$

2. a. $0 = \emptyset$ **b.** $\{x,y\} \in \{x,y,z\}$

3. a. $\{x | 0 < x < -1\} = \emptyset$
b. $\{x | 0 < x < -1\} = 0$

4. a. $\{x | x(x-1) = 0\} = \{0,1\}$
b. $\{x | x^2 + 1 < 0\} = \emptyset$

5. If $A = \{u,v,y,z\}$, determine whether the following statements are true or false.

a. $w \in A$ **b.** $x \notin A$
c. $\{u,x\} \cup A$ **d.** $\{y,z,v,u\} = A$

6. If $A = \{u, v, y, z\}$, determine whether the following statements are true or false.
a. $x \notin A$ **b.** $\{u, w\} \notin A$
c. $\{x, w\} \not\subset A$ **d.** $\emptyset \subset A$

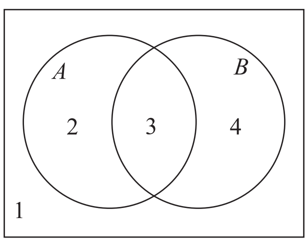
7. List all the subsets of **a.** $\{3\}$, **b.** $\{3, 4\}$.

8. List all the subsets of **a.** $\emptyset$, **b.** $\{3, 4, 5\}$.

9. Use Venn diagrams to indicate the following.
a. $A \subset U$, $B \subset U$, $A \subset B^c$
b. $A \subset U$, $B \subset U$, $B \subset A^c$

10. Use Venn diagrams to indicate the following.
a. $A \subset U$, $B \subset U$, $C \subset U$, $C \subset (A \cup B)^c$
b. $A \subset U$, $B \subset U$, $C \subset U$, $C \subset A \cap B$

For Exercises 11 through 14, indicate where the sets are located on the figure below and indicate if the sets found in part a and part b are disjoint or not.



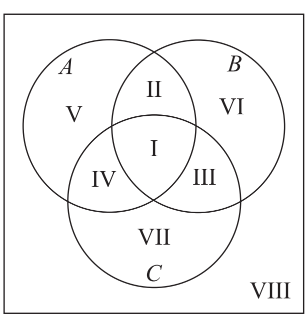
11. **a.** $A \cap B^c$ **b.** $A \cap B$

12. **a.** $A^c \cap B$ **b.** $A^c \cap B^c$

13. **a.** $A \cup B^c$ **b.** $(A \cup B)^c$

14. **a.** $A^c \cup B^c$ **b.** $(A \cap B)^c$

For Exercises 15 through 22, indicate where the sets are located on the figure below



15. **a.** $A \cap B \cap C$ **b.** $A \cap B^c \cap C^c$

16. **a.** $A \cap B \cap C^c$ **b.** $B \cap A^c \cap C^c$

17. **a.** $A^c \cap B^c \cap C^c$ **b.** $A \cap C \cap B^c$

18. **a.** $B \cap C \cap A^c$ **b.** $C \cap A^c \cap B^c$

19. **a.** $(A \cup B) \cap C^c$ **b.** $(A \cap B)^c \cap C$

20. **a.** $A \cup (B \cap C)$ **b.** $A \cup B \cup C^c$

21. **a.** $(A \cup B)^c \cap C$ **b.** $(A^c \cap B)^c \cup C$

22. **a.** $A \cup (B^c \cap C^c)$ **b.** $(A \cup B \cup C)^c \cap A$

In Exercises 23 through 30, find the indicated sets with
$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$A = \{1, 2, 3, 4, 5, 6\}$,
$B = \{4, 5, 6, 7, 8\}$, $C = \{5, 6, 7, 8, 9, 10\}$

23. **a.** $A \cap B$ **b.** $A \cup B$

24. **a.** A^c **b.** $A^c \cap B$

25. **a.** $A \cap B^c$ **b.** $A^c \cap B^c$

26. **a.** $A^c \cup B^c$ **b.** $(A^c \cup B^c)^c$

27. **a.** $A \cap B \cap C$ **b.** $(A \cap B \cap C)^c$

28. **a.** $A \cap (B \cup C)$ **b.** $A \cap (B^c \cup C)$

29. **a.** $A^c \cap B^c \cap C^c$ **b.** $(A \cup B \cup C)^c$

30. **a.** $A^c \cap B^c \cap C$ **b.** $A^c \cap B \cap C^c$

In Exercises 31 through 34, describe each of the sets in words.

Let U be the set of all residents of your state and let
$A = \{x | x \text{ owns an automobile}\}$
$H = \{x | x \text{ owns a house}\}$

31. **a.** A^c **b.** $A \cup H$ **c.** $A \cup H^c$

32. **a.** H^c **b.** $A \cap H$ **c.** $A^c \cap H$

33. **a.** $A \cap H^c$ **b.** $A^c \cap H^c$ **c.** $A^c \cup H^c$

34. **a.** $(A \cap H)^c$ **b.** $(A \cup H)^c$ **c.** $(A^c \cap H^c)^c$

In Exercises 35 through 38, let U, A, and H be as in the previous four problems, and let
$P = \{x|x \text{ owns a piano}\}$,
and describe each of the sets in words.

35. **a.** $A \cap H \cap P$ **b.** $A \cup H \cup P$
c. $(A \cap H) \cup P$

36. **a.** $(A \cup H) \cap P$ **b.** $(A \cup H) \cap P^c$
c. $A \cap H \cap P^c$

37. **a.** $(A \cap H)^c \cap P$ **b.** $A^c \cap H^c \cap P^c$
c. $(A \cup H)^c \cap P$

38. **a.** $(A \cup H \cup P)^c \cap A$ **b.** $(A \cup H \cup P)^c$
c. $(A \cap H \cap P)^c$

In Exercises 39 through 46, let U be the set of major league baseball players and let
$N = \{x|x \text{ plays for the New York Yankees}\}$
$S = \{x|x \text{ plays for the San Francisco Giants}\}$
$F = \{x|x \text{ is an outfielder}\}$
$H = \{x|x \text{ has hit 20 homers in one season}\}$
Write the set that represents the following descriptions.

39. **a.** Outfielders for the New York Yankees
b. New York Yankees who have never hit 20 homers in a season

40. **a.** San Francisco Giants who have hit 20 homers in a season.
b. San Francisco Giants who do not play outfield.

41. **a.** Major league ball players who play for the New York Yankees or the San Francisco Giants.
b. Major league ball players who play for neither the New York Yankees nor the San Francisco Giants.

42. **a.** San Francisco Giants who have never hit 20 homers in a season.
b. Major league ball players who have never hit 20 homers in a season.

43. **a.** New York Yankees or San Francisco Giants who have hit 20 homers in a season.
b. Outfielders for the New York Yankees who have never hit 20 homers in a season.

44. **a.** Outfielders for the New York Yankees or San Francisco Giants.
b. Outfielders for the New York Yankees who have hit 20 homers in a season.

45. **a.** Major league outfielders who have hit 20 homers in a season and do not play for the New York Yankees or the San Francisco Giants.
b. Major league outfielders who have never hit 20 homers in a season and do not play for the New York Yankees or the San Francisco Giants.

46. **a.** Major league players who do not play outfield, who have hit 20 homers in a season, and do not play for the New York Yankees or the San Francisco Giants.
b. Major league players who play outfield, who have never hit 20 homers in a season, and do not play for the New York Yankees or the San Francisco Giants.

In Exercises 47 through 52, let
$U = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,2,3,4,5\}$,
$B = \{4,5,6,7\}$, $C = \{5,6,7,8,9,10\}$.
Verify that the identities are true for these sets.

47. $A \cup (B \cup C) = (A \cup B) \cup C$

48. $A \cap (B \cap C) = (A \cap B) \cap C$

49. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

50. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

51. $(A \cup B)^c = A^c \cap B^c$

52. $(A \cap B)^c = A^c \cup B^c$

Solutions to Self-Help Exercises 1.1

1. **a.** $A \cup B$ is the elements in A or B. Thus $A \cup B = \{1,2,3,4,5\}$.
b. $A \cap B$ is the elements in both A and B. Thus $A \cap B = \{3,4\}$.
c. A^c is the elements not in A (but in U). Thus $A^c = \{5,6,7\}$.

d. $(A \cup B) \cap C$ is those elements in $A \cup B$ and also in C. From **a** we have

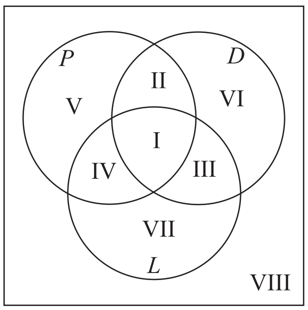
$$(A \cup B) \cap C = \{1,2,3,4,5\} \cap \{2,3,4,5,6\} = \{2,3,4,5\}$$

e. $(A \cap B) \cup C$ is those elements in $A \cap B$ or in C. Thus from **b**

$$(A \cap B) \cup C = \{3,4\} \cup \{2,3,4,5,6\} = \{2,3,4,5,6\}$$

f. $A^c \cup B \cup C$ is elements in B, or in C, or not in A. Thus

$$A^c \cup B \cup C = \{2,3,4,5,6,7\}$$



2. **a.** Corporations in this country that had profits and also paid a dividend last year is represented by $P \cap D$. This is regions I and II.

b. Corporations in this country that either had profits or paid a dividend last year is represented by $P \cup D$. This is regions I, II, III, IV, V, and VI.

c. Corporations in this country that did not have profits is represented by P^c. This is regions III, VI, VII, and VIII.

d. Corporations in this country that had profits, paid a dividend, and did not increase their labor force last year is represented by $P \cap D \cap L^c$. This is region II.

e. Corporations in this country that had profits or paid a dividend, and did not increase their labor force last year is represented by $(P \cup D) \cap L^c$. This is regions II, V, and VI.

1.2 The Number of Elements in a Set

APPLICATION
Breakfast Survey

In a survey of 120 adults, 55 said they had an egg for breakfast that morning, 40 said they had juice for breakfast, and 70 said they had an egg or juice for breakfast. How many had an egg but no juice for breakfast? How many had neither an egg nor juice for breakfast? See Example 1 for the answer.

✧ Counting the Elements of a Set

This section shows the relationship between the number of elements in $A \cup B$ and the number of elements in A, B, and $A \cap B$. This is our first counting principle. The examples and exercises in this section give some applications of this. In other applications we will count the number of elements in various sets to find probability.

The Notation $n(A)$
If A is a set with a finite number of elements, we denote the number of elements in A by $n(A)$.

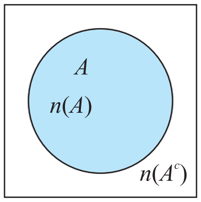


Figure 1.9

In Figure 1.9 we see the number $n(A)$ written inside the A circle and $n(A^c)$ written outside the set A. This indicates that there are $n(A)$ members in set A and $n(A^c)$ in set A^c. The number of elements in a set is also called the **cardinality** of the set.

There are two results that are rather apparent. First, the empty set $\emptyset$ has no elements $n(\emptyset) = 0$. For the second refer to Figure 1.10 where the two sets A and B are disjoint.

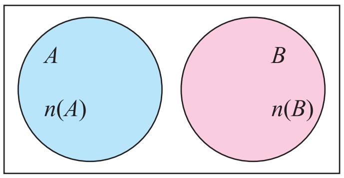


Figure 1.10

The Number in the Union of Disjoint Sets

If the sets A and B are disjoint, then

$$n(A \cup B) = n(A) + n(B)$$

A consequence of the last result is the following. In Figure 1.9, we are given a universal set U and a set $A \subset U$. Then since $A \cap A^c = \emptyset$ and $U = A \cup A^c$,

$$n(U) = n(A \cup A^c) = n(A) + n(A^c)$$

✧ Union Rule for Two Sets

Now consider the more general case shown in Figure 1.11. We assume that x is the number in the set A that are not in B, that is, $n(A \cap B^c)$. Next we have z, the number in the set B that are not in A, $n(A^c \cap B)$. Finally, y is the number in both A and B, $n(A \cap B)$ and w is the number of elements that are neither in A nor in B, $n(A^c \cap B^c)$. Then

$$\begin{aligned} n(A \cup B) &= x + y + z \\ &= (x + y) + (y + z) - y \\ &= n(A) + n(B) - n(A \cap B) \end{aligned}$$

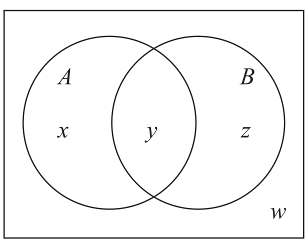


Figure 1.11

Alternatively, we can see that the total $n(A) + n(B)$ counts the number in the intersection $n(A \cap B)$ twice. Thus to obtain the number in the union $n(A \cup B)$, we must subtract $n(A \cap B)$ from $n(A) + n(B)$.

The Number in the Union of Two Sets

For any finite sets A and B,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

EXAMPLE 1 **An Application of Counting** In a survey of 120 adults, 55 said they had an egg for breakfast that morning, 40 said they had juice for breakfast, and 70 said they had an egg or juice for breakfast. How many had an egg but no juice for breakfast? How many had neither an egg nor juice for breakfast?

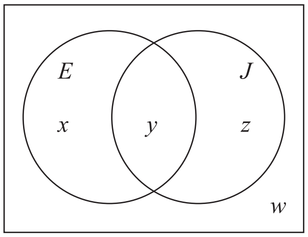


Figure 1.12a

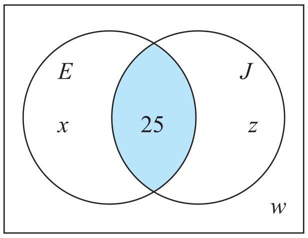


Figure 1.12b

Solution Let U be the universal set of adults surveyed, E the set that had an egg for breakfast, and J the set that had juice for breakfast. A Venn diagram is shown in Figure 1.12a. From the survey, we have that

$$n(E) = 55 \qquad n(J) = 40, \qquad n(E \cup J) = 70$$

Note that each of these is a sum. That is $n(E) = 55 = x + y$, $n(J) = 40 = y + z$ and $n(E \cup J) = 70 = x + y + z$. Since 120 people are in the universal set, $n(U) = 120 = x + y + z + w$.

The number that had an egg and juice for breakfast is given by $n(E \cap J)$ and is shown as the shaded region in Figure 1.12b. We apply the union rule:

$$\begin{aligned} n(E \cap J) &= n(E) + n(J) - n(E \cup J) \\ &= 55 + 40 - 70 \\ &= 25 \end{aligned}$$

We first place the number 25, just found, in the $E \cap J$ area in the Venn diagram in Figure 1.12b. Since the number of people who had eggs (with and without juice) is 55, then according to Figure 1.12b,

$$\begin{aligned} n(E) = 55 &= x + 25 \\ x &= 30 \end{aligned}$$

Similarly, the number who had juice (with and without an egg) is 40. Using Figure 1.12b,

$$\begin{aligned} n(J) = 40 &= z + 25 \\ z &= 15 \end{aligned}$$

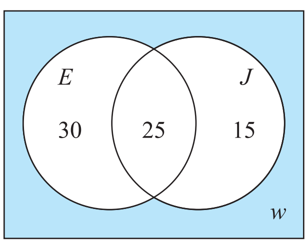


Figure 1.12c

These two results are shown in Figure 1.12c. We wish to find $w = n((E \cup J)^c)$. This is shown as the shaded region in Figure 1.12c. The unshaded region is $E \cup J$. We then have that

$$\begin{aligned} n(E \cup J) + n((E \cup J)^c) &= n(U) \\ n((E \cup J)^c) &= n(U) - n(E \cup J) \\ w &= 120 - 70 \\ w &= 50 \end{aligned}$$

And so there were 50 people in the surveyed group that had neither an egg nor juice for breakfast.

Counting With Three Sets

Many counting problems with sets have two sets in the universal set. We will also study applications with three sets in the universal set. The union rule for three sets is studied in the extensions for this section. In the example below, deductive reasoning is used to solve for the number of elements in each region of the Venn diagram. In cases where this will not solve the problem, systems of linear equations can be used to solve the Venn diagram. This is studied in the Chapter Project found in the Review section.

EXAMPLE 2 **European Travels** In a survey of 200 people that had just returned from a trip to Europe, the following information was gathered.

- 142 visited England
- 95 visited Italy
- 65 visited Germany
- 70 visited both England and Italy
- 50 visited both England and Germany
- 30 visited both Italy and Germany
- 20 visited all three of these countries

a. How many went to England but not Italy or Germany?
b. How many went to exactly one of these three countries?
c. How many went to none of these three countries?

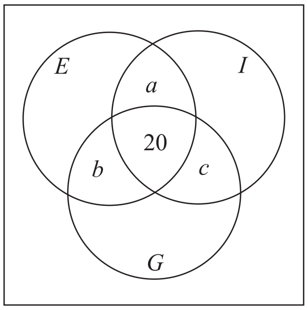


Figure 1.13a

Solution Let U be the set of 200 people that were surveyed and let
$E = \{x|x \text{ visited England}\}$
$I = \{x|x \text{ visited Italy}\}$
$G = \{x|x \text{ visited Germany}\}$

We first note that the last piece of information from the survey indicates that

$$n(E \cap I \cap G) = 20$$

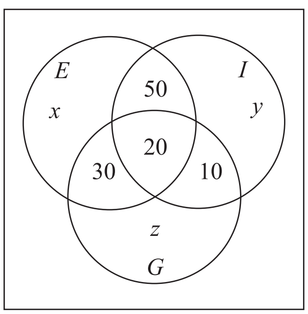


Figure 1.13b

Place this in the Venn diagram shown in Figure 1.13a. Recall that 70 visited both England and Italy, that is, $n(E \cap I) = 70$. If a is the number that visited England and Italy but not Germany, then, according to Figure 1.13a, $20 + a = n(E \cap I) = 70$. Thus $a = 50$. In the same way, if b is the number that visited England and Germany but not Italy, then $20 + b = n(E \cap G) = 50$. Thus $b = 30$. Also if c is the number that visited Italy and Germany but not England, then $20 + c = n(G \cap I) = 30$. Thus $c = 10$. All of this information is then shown in Figure 1.13b.

a. Let x denote the number that visited England but not Italy or Germany. Then, according to Figure 1.13b, $20 + 30 + 50 + x = n(E) = 142$. Thus $x = 42$, that is, the number that visited England but not Italy or Germany is 42.

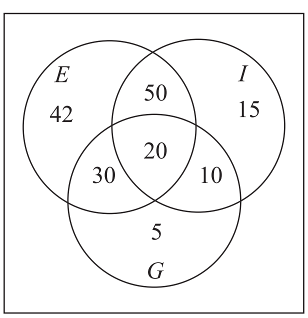


Figure 1.13c

b. Since $n(I) = 95$, the number that visited Italy but not England or Germany is given from Figure 1.13b by $95 - (50 + 20 + 10) = 15$. Since $n(G) = 65$, the number that visited Germany but not England or Italy is, according to Figure 1.13b, given by $65 - (30 + 20 + 10) = 5$. Thus, according to Figure 1.13c, the number who visited just one of the three countries is

$$42 + 15 + 5 = 62$$

c. There are 200 people in the U and so according to Figure 1.13c, the number that visited none of these three countries is given by

$$200 - (42 + 15 + 5 + 50 + 30 + 10 + 20) = 200 - 172 = 28$$ ✦

EXAMPLE 3 **Pizzas** At the end of the day the manager of Blue Baker wanted to know how many pizzas were sold. The only information he had is listed below. Use the information to determine how many pizzas were sold.

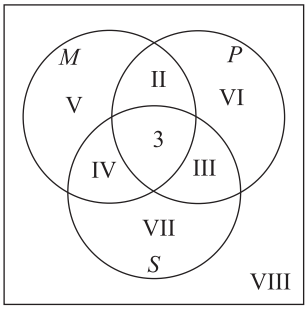


Figure 1.14a

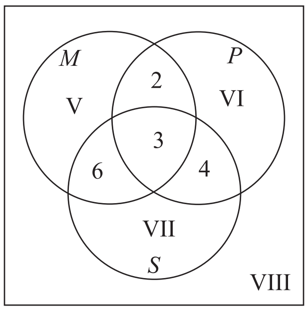


Figure 1.14b

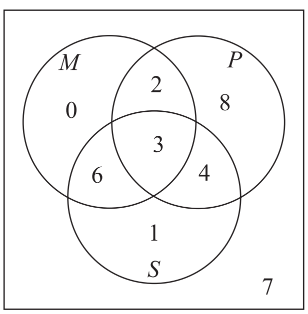


Figure 1.14c

- 3 pizzas had mushrooms, pepperoni, and sausage
- 7 pizzas had pepperoni and sausage
- 6 pizzas had mushrooms and sausage but not pepperoni
- 15 pizzas had two or more of these toppings
- 11 pizzas had mushrooms
- 8 pizzas had only pepperoni
- 24 pizzas had sausage or pepperoni
- 17 pizzas did not have sausage

Solution Begin by drawing a Venn diagram with a circle for pizzas that had mushrooms, a circle for pizzas that had pepperoni, and, pizzas that had sausage. In the center place a 3 since three pizzas had all these toppings. See Figure 1.14a.

Since 7 pizzas have pepperoni and sausage, 7 = 3 + III or III = 4. If 6 pizzas had mushrooms and sausage but not pepperoni, then IV = 6. The region for two or more of these toppings is 3 + II + III + IV = 15. Using III = 4 and IV = 6, that gives 3 + II + 4 + 6 = 15 or II = 2. This information is shown in Figure 1.14b.

Given that 11 pizzas had mushrooms, V + 2 + 3 + 6 = 11 and therefore V = 0. Since 8 pizzas had only pepperoni, VI = 8. With a total of 24 pizzas in the sausage or pepperoni region and knowing that VI = 8, we have 2 + 8 + 6 + 3 + 4 + VII = 24 or VII = 1. Finally, if 17 pizzas did not have sausage then 17 = V + 2 + VI + VIII = 0 + 2 + 8 + VIII. This gives VIII = 7 and our complete diagram is shown in Figure 1.14c.

To find the total number of pizzas sold, the 8 numbers in the completed Venn diagram are added:

$$0+2+8+6+3+4+1+7=31$$

✦

Self-Help Exercises 1.2

1. Given that $n(A\cup B)=100$, $n(A\cap B^c)=50$, and $n(A\cap B)=20$, find $n(A^c\cap B)$.

2. The registrar reported that among 2000 students, 700 did not register for a math or English course, while 400 registered for both of these two courses. How many registered for exactly one of these courses?

3. One hundred shoppers are interviewed about the contents of their bags and the following results are found:

- 40 bought apple juice
- 19 bought cookies
- 13 bought broccoli
- 1 bought broccoli, apple juice, and cookies
- 11 bought cookies and apple juice
- 2 bought cookies and broccoli but not apple juice
- 24 bought only apple juice

Organize this information in a Venn diagram and find how many shoppers bought none of these items.

1.2 Exercises

1. If $n(A) = 100$, $n(B) = 75$, and $n(A \cap B) = 40$, what is $n(A \cup B)$?

2. If $n(A) = 200$, $n(B) = 100$, and $n(A \cup B) = 250$, what is $n(A \cap B)$?

3. If $n(A) = 100$, $n(A \cap B) = 20$, and $n(A \cup B) = 150$, what is $n(B)$?

4. If $n(B) = 100$, $n(A \cup B) = 175$, and $n(A \cap B) = 40$, what is $n(A)$?

5. If $n(A) = 100$ and $n(A \cap B) = 40$, what is $n(A \cap B^c)$?

6. If $n(U) = 200$ and $n(A \cup B) = 150$, what is $n(A^c \cap B^c)$?

7. If $n(A \cup B) = 500$, $n(A \cap B^c) = 200$, $n(A^c \cap B) = 150$, what is $n(A \cap B)$?

8. If $n(A \cap B) = 50$, $n(A \cap B^c) = 200$, $n(A^c \cap B) = 150$, what is $n(A \cup B)$?

9. If $n(A \cap B) = 150$ and $n(A \cap B \cap C) = 40$, what is $n(A \cap B \cap C^c)$?

10. If $n(A \cap C) = 100$ and $n(A \cap B \cap C) = 60$, what is $n(A \cap B^c \cap C)$?

11. If $n(A) = 200$ and $n(A \cap B \cap C) = 40$, $n(A \cap B \cap C^c) = 20$, $n(A \cap B^c \cap C) = 50$, what is $n(A \cap B^c \cap C^c)$?

12. If $n(B) = 200$ and $n(A \cap B \cap C) = 40$, $n(A \cap B \cap C^c) = 20$, $n(A^c \cap B \cap C) = 50$, what is $n(A^c \cap B \cap C^c)$?

For Exercises 13 through 20, let A, B, and C be sets in a universal set U. We are given $n(U) = 100$, $n(A) = 40$, $n(B) = 37$, $n(C) = 35$, $n(A \cap B) = 25$, $n(A \cap C) = 22$, $n(B \cap C) = 24$, and $n(A \cap B \cap C^c) = 10$. Find the following values.

13. $n(A \cap B \cap C)$

14. $n(A^c \cap B \cap C)$

15. $n(A \cap B^c \cap C)$

16. $n(A \cap B^c \cap C^c)$

17. $n(A^c \cap B \cap C^c)$

18. $n(A^c \cap B^c \cap C)$

19. $n(A \cup B \cup C)$

20. $n((A \cup B \cup C))^c$

Applications

21. **Headache Medicine** In a survey of 1200 households, 950 said they had aspirin in the house, 350 said they had acetaminophen, and 200 said they had both aspirin and acetaminophen.
    - **a.** How many in the survey had at least one of the two medications?
    - **b.** How many in the survey had aspirin but not acetaminophen?
    - **c.** How many in the survey had neither aspirin nor acetaminophen?

22. **Newspaper Subscriptions** In a survey of 1000 households, 600 said they received the morning paper but not the evening paper, 300 said they received both papers, and 100 said they received neither paper.
    - **a.** How many received the evening paper but not the morning paper?
    - **b.** How many received at least one of the papers?

23. **Course Enrollments** The registrar reported that among 1300 students, 700 students did not register for either a math or English course, 400 registered for an English course, and 300 registered for both types of courses.
    - **a.** How many registered for an English course but not a math course?
    - **b.** How many registered for a math course?

24. **Pet Ownership** In a survey of 500 people, a pet food manufacturer found that 200 owned a dog but not a cat, 150 owned a cat but not a dog, and 100 owned neither a dog or cat.
    - **a.** How many owned both a cat and a dog?
    - **b.** How many owned a dog?

25. **Fast Food** A survey by a fast-food chain of 1000 adults found that in the past month 500 had been to Burger King, 700 to McDonald's, 400 to Wendy's, 300 to Burger King and McDonald's, 250 to McDonald's and Wendy's, 220 to Burger King and Wendy's, and 100 to all three. How many went to
    - **a.** Wendy's but not the other two?
    - **b.** only one of them?
    - **c.** none of these three?

26. Investments A survey of 600 adults over age 50 found that 200 owned some stocks and real estate but no bonds, 220 owned some real estate and bonds but no stock, 60 owned real estate but no stocks or bonds, and 130 owned both stocks and bonds. How many owned none of the three?

27. Entertainment A survey of 500 adults found that 190 played golf, 200 skied, 95 played tennis, 100 played golf but did not ski or play tennis, 120 skied but did not play golf or tennis, 30 played golf and skied but did not play tennis, and 40 did all three.

a. How many played golf and tennis but did not ski?

b. How many played tennis but did not play golf or ski?

c. How many participated in at least one of the three sports?

28. Transportation A survey of 600 adults found that during the last year, 100 traveled by plane but not by train, 150 traveled by train but not by plane, 120 traveled by bus but not by train or plane, 100 traveled by both bus and plane, 40 traveled by all three, and 360 traveled by plane or train. How many did not travel by any of these three modes of transportation?

29. Magazines In a survey of 250 business executives, 40 said they did not read Money, Fortune, or Business Week, while 120 said they read exactly one of these three and 60 said they read exactly two of them. How many read all three?

30. Sales A furniture store held a sale that attracted 100 people to the store. Of these, 57 did not buy anything, 9 bought both a sofa and love seat, 8 bought both a sofa and chair, 7 bought both a love seat and chair. There were 24 sofas, 18 love seats, and 20 chairs sold. How many people bought all three items?

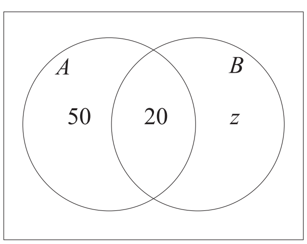
Extensions

31. Use a Venn diagram to show that

$$\begin{aligned} n(A\cup B\cup C) = n(A)+n(B)+n(C) \\ -n(A\cap B)-n(A\cap C) \\ -n(B\cap C)+n(A\cap B\cap C) \end{aligned}$$

32. Give a proof of the formula in Exercise 31. Hint: Set $B\cup C = D$ and use union rule on $n(A\cup D)$. Now use the union rule two more times, recalling from the last section that $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$.

Solutions to Self-Help Exercises 1.2



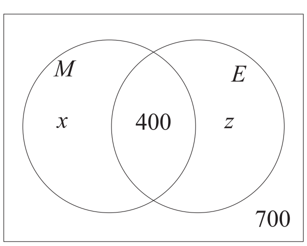
1. The accompanying Venn diagram indicates that

$$n(A\cap B^c) = 50, \quad n(A\cap B) = 20, \quad z = n(A^c\cap B)$$

Then, according to the diagram,

$$50+20+z = n(A\cup B) = 100$$

Thus $z = 30$.



2. The number of students that registered for exactly one of the courses is the number that registered for math but not English, $x = n(M\cap E^c)$, plus the number that registered for English but not math, $z = n(M^c\cap E)$. Then, according to the accompanying Venn diagram, $x+z+400+700 = 2000$. Thus $x+z = 900$. That is, 900 students registered for exactly one math or English course.

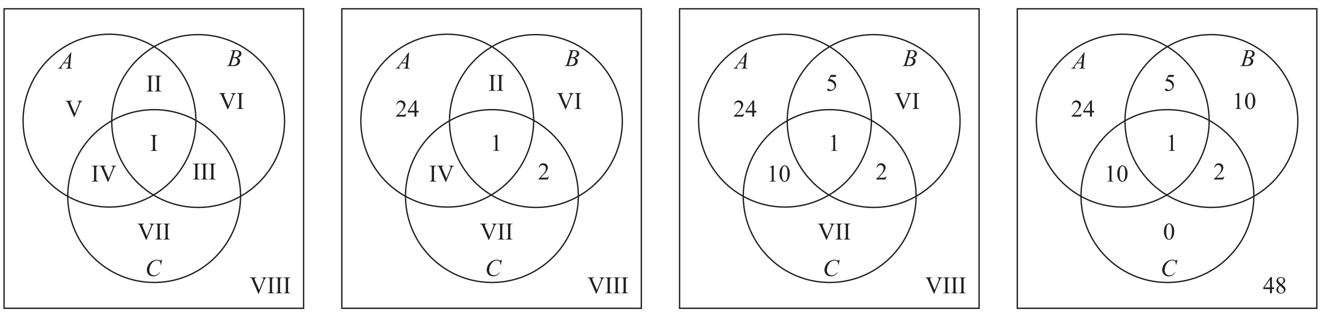
3. Let A be the set of shoppers who bought apple juice, B the set of shoppers who bought broccoli, and C the set of shoppers who bought cookies. This is shown in the first figure below. Since one shopper bought all three items, a 1 is placed in region I. Twenty-four shoppers bought only apple juice and this is region V.

Given 2 shoppers bought cookies and broccoli but not apple juice, a 2 is placed in region III. This is shown in the next figure below.

The statement "11 bought cookies and apple juice" includes those who bought broccoli and those who did not. We now know that one person bought all 3 items, so $11 - 1 = 10$ people bought cookies and apple juice but not broccoli. A 10 is placed in region IV.

Now I + II + IV + V = 40 as we are told "40 bought apple juice." With the 10 in region IV we know 3 of the 4 values for set A and we can solve for region II: 40 = 24 + 10 + 1 + II gives II = 5. Place this in the Venn diagram as shown in the third figure below.

Examining the figure, we can use the total of 13 in the broccoli circle to solve for VI: 18 = 5 + 1 + 2 + VI gives VI = 10. The total of 19 in the cookies circle lets us solve for VII: 10 + 1 + 2 + VII = 13 gives VII = 0. The very last piece of information is that there were 100 shoppers. To solve for VIII we have 100 = 24 + 5 + 10 + 10 + 1 + 2 + 0 + VIII or VIII = 48. That is, 48 shoppers bought none of these items. The completed diagram is the final figure below.



1.3 Sample Spaces and Events

Many people have a good idea of the basics of probability. That is, if a fair coin is flipped, you have an equal chance of a head or a tail showing. However, as we proceed to study more advanced concepts in probability we need some formal definitions that will both agree with our intuitive understanding of probability and allow us to go deeper into topics such as conditional probability. This will tie closely to work we have done learning about sets.

✧ The Language of Probability

We begin the preliminaries by stating some definitions. It is very important to have a clear and precise language to discuss probability so pay close attention to the exact meanings of the terms below.

Experiments and Outcomes
An **experiment** is an activity that has observable results.
An **outcome** is the result of the experiment.

The following are some examples of experiments. Flip a coin and observe whether it falls "heads" or "tails." Throw a die (a small cube marked on each face with from one to six dots[1]) and observe the number of dots on the top face. Select a transistor from a bin and observe whether or not it is defective.

The following are some additional terms that are needed.

> **Sample Spaces and Trials**
> A **sample space** of an experiment is the set of all possible outcomes of the experiment. Each repetition of an experiment is called a **trial**.

For the experiment of throwing a die and observing the number of dots on the top face the sample space is the set

$$S = \{1, 2, 3, 4, 5, 6\}$$

In the experiment of flipping a coin and observing whether it falls heads or tails, the sample space is $S = \{\text{heads}, \text{tails}\}$ or simply $S = \{H, T\}$.

EXAMPLE 1 Determining the Sample Space An experiment consists of noting whether the price of the stock of the Ford Corporation rose, fell, or remained unchanged on the most recent day of trading. What is the sample space for this experiment?

Solution There are three possible outcomes depending on whether the price rose, fell, or remained unchanged. Thus the sample space S is

$$S = \{\text{rose}, \text{fell}, \text{unchanged}\}$$

✦

.

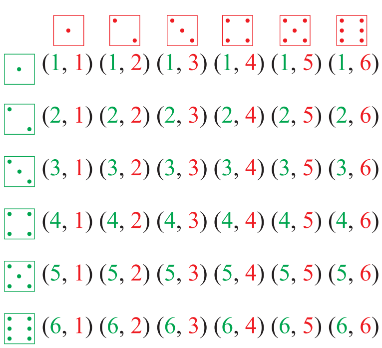


Figure 1.15

EXAMPLE 2 Determining the Sample Space Two dice, identical except that one is green and the other is red, are tossed and the number of dots on the top face of each is observed. What is the sample space for this experiment?

Solution Each die can take on its six different values with the other die also taking on all of its six different values. We can express the outcomes as order pairs. For example, (2, 3) will mean 2 dots on the top face of the green die and 3 dots on the top face of the red die. The sample space S is below. A more colorful version is shown in Figure 1.15.

$$\begin{aligned} S = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \end{aligned}$$

✦

[1]There are also four-sided die, eight-sided die, and so on. However, the six-sided die is the most common and six-sided should be assumed when we refer to a die, unless otherwise specified.

If the experiment of tossing 2 dice consists of just observing the total number of dots on the top faces of the two dice, then the sample space would be

$$S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

In short, the sample space depends on the precise statement of the experiment.

EXAMPLE 3 Determining the Sample Space A coin is flipped twice to observe whether heads or tails shows; order is important. What is the sample space for this experiment?

Solution The sample space S consists of the 4 outcomes

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$ ✦

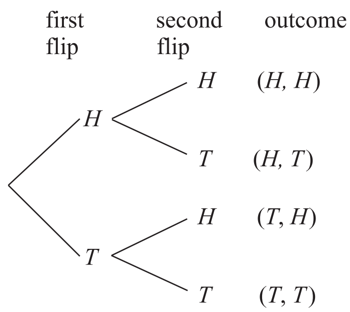


Figure 1.16

✧ Tree Diagrams

In Example 3 we completed a task (flipped a coin) and then completed another task (flipped the coin again). In these cases, the experiment can be diagrammed with a tree. The **tree diagram** for Example 3 is shown in Figure 1.16. We see we have a first set of branches representing the first flip of the coin. From there we flip the coin again and have a second set of branches. Then trace along each branch to find the outcomes of the experiment. If the coin is tossed a third time, there will be eight outcomes.

EXAMPLE 4 Determining the Sample Space A die is rolled. If the die shows a 1 or a 6, a coin is tossed. What is the sample space for this experiment?

Solution Figure 1.17 shows the possibilities. We then have

$$S = \{(1,H), (1,T), 2, 3, 4, 5, (6,H), (6,T)\}$$ ✦

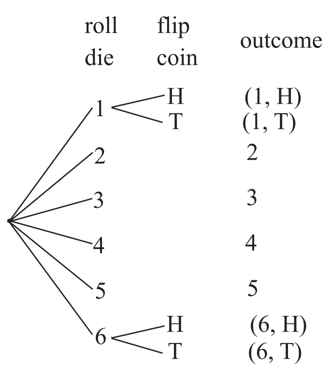


Figure 1.17

✧ Events

We start this subsection with the following definition of an event.

> **Events and Elementary Events**
> Given a sample space S for an experiment, an **event** is any subset E of S. An **elementary (or simple) event** is an event with a single outcome.

EXAMPLE 5 Finding Events Using the sample space from Example 3 find the events: "At least one head comes up" and "Exactly two tails come up." Are either events elementary events?

Solution "At least one head comes up" $= \{(H,H), (H,T), (T,H)\}$

"Exactly two tails come up" $= \{(T,T)\}$

The second event, "Exactly two tails come up" has only one outcome and so it is an elementary event. ✦

We can use our set language for union, intersection, and complement to describe events.

Union of Two Events
If E and F are two events, then $E \cup F$ is the union of the two events and consists of the set of outcomes that are in E or F.

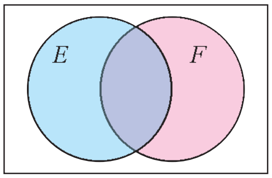


Figure 1.18

Thus the event $E \cup F$ is the event that "E or F occurs." Refer to Figure 1.18 where the event $E \cup F$ is the shaded region on the Venn diagram.

Intersection of Two Events
If E and F are two events, then $E \cap F$ is the intersection of the two events and consists of the set of outcomes that are in both E and F.

Thus the event $E \cap F$ is the event that "E and F both occur." Refer to Figure 1.18 where the event $E \cap F$ is the region where E and F overlap.

Complement of an Event
If E is an event, then E^c is the complement of E and consists of the set of outcomes that are not in E.

Thus the event E^c is the event that "E does not occur."

EXAMPLE 6 **Determining Union, Intersection, and Complement** Consider the sample space given in Example 2. Let E consist of those outcomes for which the number of dots on the top faces of both dice is 2 or 4. Let F be the event that the sum of the number of dots on the top faces of the two dice is 6. Let G be the event that the sum of the number of dots on the top faces of the two dice is less than 11.
a. List the elements of E and F.
b. Find $E \cup F$.
c. Find $E \cap F$.
d. Find G^c.

Solution
a. $E = \{(2,2),(2,4),(4,2),(4,4)\}$ and $F = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$
b. $E \cup F = \{(2,2),(2,4),(4,2),(4,4),(1,5),(3,3),(5,1)\}$
c. $E \cap F = \{(2,4),(4,2)\}$
d. $G^c = \{(5,6),(6,5),(6,6)\}$ ✦

If S is a sample space, $\emptyset \subseteq S$, and thus $\emptyset$ is an event. We call the event $\emptyset$ the **impossible event** since the event $\emptyset$ means that no outcome has occurred, whereas, in any experiment some outcome *must* occur.

The Impossible Event
The empty set, $\emptyset$, is called the **impossible event**.

For example, if H is the event that a head shows on flipping a coin and T is the event that a tail shows, then $H \cap T = \emptyset$. The event $H \cap T$ means that both heads and tails shows, which is impossible.

Since $S \subseteq S$, S is itself an event. We call S the **certainty event** since any outcome of the experiment must be in S. For example, if a fair coin is flipped, the event $H \cup T$ is certain since a head or tail must occur.

The Certainty Event
Let S be a sample space. The event S is called the **certainty event**.

We also have the following definition for mutually exclusive events. See Figure 1.19.

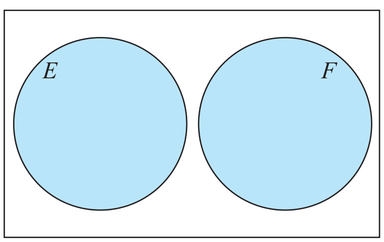


Figure 1.19

Mutually Exclusive Events
Two events E and F are said to be **mutually exclusive** if the sets are **disjoint**. That is,

$$E \cap F = \emptyset$$

Standard Deck of 52 Playing Cards
A standard deck of 52 playing cards has four 13-card suits: clubs ♣, diamonds ♢, hearts ♡, and spades ♠. The diamonds and hearts are red, while the clubs and spades are black. Each 13-card suit contains cards numbered from 2 to 10, a jack, a queen, a king, and an ace. The jack, queen, king, and ace can be considered respectively as number 11, 12, 13, and 14. In poker the ace can be either a 14 or a 1. See Figure 1.20.

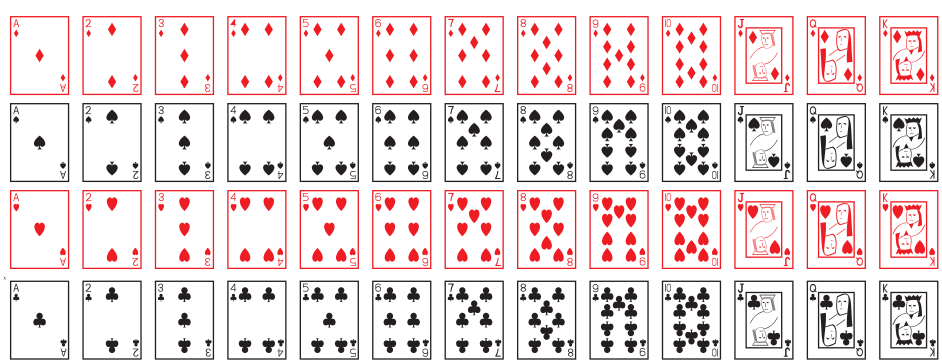


Figure 1.20

EXAMPLE 7 **Determining if Sets Are Mutually Exclusive** Let a card be chosen from a standard deck of 52 cards. Let E be the event consisting of drawing a 3. Let F be the event of drawing a heart. Let G be the event of drawing a Jack. Are E and F mutually exclusive? Are E and G?

Solution Since $E \cap F$ is the event that the card is a 3 and is a heart,

$$E \cap F = \{3\heartsuit\} \neq \emptyset$$

and so these events are not mutually exclusive. The event $E \cap G$ is the event that the card is a 3 and a jack, so

$$E \cap G = \emptyset$$

therefore E and G are mutually exclusive. ✦

✧ Continuous Sample Spaces

In all the previous examples we were able to list each outcome in the sample space, even if the list is rather long. But consider the outcomes of an experiment where the time spent running a race is measured. Depending on how the time is measured, an outcome could be 36 seconds or 36.0032 seconds. The values of the outcomes are not restricted to whole numbers and so the sample space must be described, rather than listed. In the case of the race we could say $S = \{t|t \geq 0, t \text{ in seconds}\}$. Then the event E that a person takes less than 35 seconds to run the race would be written $E = \{t|t < 35 \text{ seconds}\}$.

EXAMPLE 8 **Weighing Oranges** At a farmer's market there is a display of fresh oranges. The oranges are carefully weighed. What is a sample space for this experiment? Describe the event that a orange weighs 100 grams or more. Describe the event that a orange weighs between 200 and 250 grams.

Solution Since the weight of the orange can be any positive number,

$$S = \{w|w > 0, w \text{ in grams}\}$$

Note that $w = 0$ is not included as if the weight was zero there would be no orange! The event that an orange weighs 100 grams or more is

$$E = \{w|w \geq 100, w \text{ in grams}\}$$

Here note that we use $\geq$, not $>$ as the value of exactly 100 grams needs to be included. The event that the orange weighs between 200 and 250 grams is

$$F = \{w|200 < w < 250, w \text{ in grams}\}$$

where strict inequalities are used as the weight is between those values. ✦

Self-Help Exercises 1.3

1. Two tetrahedrons (4 sided), each with equal sides numbered from 1 to 4, are identical except that one is red and the other green. If the two tetrahedrons are tossed and the number on the bottom face of each is observed, what is the sample space for this experiment?

2. Consider the sample space given in the previous exercise. Let E consist of those outcomes for which both (tetrahedron) dice show an odd number. Let F be the event that the sum of the two numbers on these dice is 5. Let G be the event that the sum of the two numbers is less than 7.

   a. List the elements of E and F.

   b. Find $E \cap F$.

   c. Find $E \cup F$.

   d. Find G^c.

3. A hospital carefully measures the length of every baby born. What is a sample space for this experiment? Describe the events

   a. the baby is longer than 22 inches.

   b. the baby is 20 inches or shorter.

   c. the baby is between 19.5 and 21 inches long.

1.3 Exercises

1. Let $S = \{a, b, c\}$ be a sample space. Find all the events.

2. Let the sample space be $S = \{a, b, c, d\}$. How many events are there?

3. A coin is flipped three times, and heads or tails is observed after each flip. What is the sample space? Indicate the outcomes in the event "at least 2 heads are observed."

4. A coin is flipped, and it is noted whether heads or tails show. A die is tossed, and the number on the top face is noted. What is the sample space of this experiment?

5. A coin is flipped three times. If heads show, one is written down. If tails show, zero is written down. What is the sample space for this experiment? Indicate the outcomes if "one is observed at least twice."

6. Two tetrahedrons (4 sided), each with equal sides numbered from 1 to 4, are identical except that one is red and the other green. If the two tetrahedrons are tossed and the number on the bottom face of each is observed, indicate the outcomes in the event "the sum of the numbers is 4."

7. An urn holds 10 identical balls except that 1 is white, 4 are black, and 5 are red. An experiment consists of selecting a ball from the urn and observing its color. What is a sample space for this experiment? Indicate the outcomes in the event "the ball is not white."

8. For the urn in Exercise 7, an experiment consists of selecting 2 balls in succession without replacement and observing the color of each of the balls. What is the sample space of this experiment? Indicate the outcomes of the event "no ball is white."

9. Ann, Bubba, Carlos, David, and Elvira are up for promotion. Their boss must select three people from this group of five to be promoted. What is the sample space? Indicate the outcomes of the event "Bubba is selected."

10. A restaurant offers six side dishes: rice, macaroni, potatoes, corn, broccoli, and carrots. A customer must select two different side dishes for his dinner. What is the sample space? List the outcomes of the event "Corn is selected."

11. An experiment consists of selecting a digit from the number 112964333 and observing it. What is a sample space for this experiment? Indicate the outcomes in the event that "an even digit."

12. An experiment consists of selecting a letter from the word CONNECTICUT and observing it. What is a

sample space for this experiment? Indicate the outcomes of the event “a vowel is selected.”

13. An inspector selects 10 transistors from the production line and notes how many are defective.

- **a.** Determine the sample space.
- **b.** Find the outcomes in the set corresponding to the event E “at least 6 are defective.”
- **c.** Find the outcomes in the set corresponding to the event F “at most 4 are defective.”
- **d.** Find the sets $E \cup F$, $E \cap F$, E^c, $E \cap F^c$, $E^c \cap F^c$.
- **e.** Find all pairs of sets among the nonempty ones listed in part (d) that are mutually exclusive.

14. A survey indicates first whether a person is in the lower income group (L), middle income group (M), or upper income group (U), and second which of these groups the father of the person is in.

- **a.** Determine the sample space using the letters L, M, and U.
- **b.** Find the outcomes in the set corresponding to the event E “the person is in the lower income group.”
- **c.** Find the outcomes in the set corresponding to the event F “the person is in the higher income group.”
- **d.** Find the sets $E \cup F$, $E \cap F$, E^c, $E \cap F^c$, $E^c \cap F^c$.
- **e.** Find all pairs of sets listed in part (d) that are mutually exclusive.

15. A corporate president decides that for each of the next three fiscal years success (S) will be declared if the earnings per share of the company go up at least 10% that year and failure (F) will occur is less than 10%.

- **a.** Determine the sample space using the letters S and F.
- **b.** Find the outcomes in the set corresponding to the event E “at least 2 of the next 3 years is a success.”
- **c.** Find the outcomes in the set corresponding to the event G “the first year is a success.”
- **d.** Find and describe the sets $E \cup G$, $E \cap G$, G^c, $E^c \cap G$, and $(E \cup G)^c$.
- **e.** Find all pairs of sets listed in part (d) that are mutually exclusive.

16. Let E be the event that the life of a certain light bulb is at least 100 hours and F that the life is at most 200 hours. Describe the sets:

a. $E \cap F$ **b.** F^c **c.** $E^c \cap F$ **d.** $(E \cup F)^c$

17. Let E be the event that a pencil is 10 cm or longer and F the event that the pencil is less than 25 cm. Describe the sets:

a. $E \cap F$ **b.** E^c **c.** $E \cap F^c$ **d.** $(E \cup F)^c$

In Exercises 18 through 23, S is a sample space and E, F, and G are three events. Use the symbols $\cap$, $\cup$, and ^{c} to describe the given events.

18. F but not E

19. E but not F

20. Not F or not E

21. Not F and not E

22. Not F, nor E, nor G

23. E and F but not G

24. Let S be a sample space consisting of all the integers from 1 to 20 inclusive, E the first 10 of these, and F the last 5 of these. Find $E \cap F$, $E^c \cap F$, $(E \cup F)^c$, and $E^c \cap F^c$.

25. Let S be the 26 letters of the alphabet, E be the vowels $\{a, e, i, o, u\}$, F the remaining 21 letters, and G the first 5 letters of the alphabet. Find the events $E \cup F \cup G$, $E^c \cup F^c \cup G^c$, $E \cap F \cap G$, and $E \cup F^c \cup G$.

26. A bowl contains a penny, a nickel, and a dime. A single coin is chosen at random from the bowl. What is the sample space for this experiment? List the outcomes in the event that a penny or a nickel is chosen.

27. A cup contains four marbles. One red, one blue, one green, and one yellow. A single marble is drawn at random from the cup. What is the sample space for this experiment? List the outcomes in the event that a blue or a green marble is chosen.

Solutions to Self-Help Exercises 1.3

1. Consider the outcomes as ordered pairs, with the number on the bottom of the red one the first number and the number on the bottom of the white one the second number. The sample space is

$$\begin{aligned} S = \{ &(1,1),(1,2),(1,3),(1,4),\\ &(2,1),(2,2),(2,3),(2,4),\\ &(3,1),(3,2),(3,3),(3,4),\\ &(4,1),(4,2),(4,3),(4,4)\} \end{aligned}$$

2. **a.** $E = \{(1,1),(1,3),(3,1),(3,3)\}$, and $F = \{(1,4),(2,3),(3,2),(4,1)\}$

   b. $E \cap F = \emptyset$

   c. $E \cup F = \{(1,1),(1,3),(3,1),(3,3),(1,4),(2,3),(3,2),(4,1)\}$

   d. $G^c = \{(3,4),(4,3),(4,4)\}$

3. Since the baby can be any length greater than zero, the sample space is

$$S = \{x | x > 0, x \text{ in inches}\}$$

   a. $E = \{x | x > 22, x \text{ in inches}\}$

   b. $F = \{x | x \leq 20, x \text{ in inches}\}$

   c. $G = \{x | 19.5 < x < 21, x \text{ in inches}\}$

1.4 Basics of Probability

✧ Introduction to Probability

We first consider sample spaces for which the outcomes (elementary events) are equally likely. For example, a head or tail is equally likely to come up on a flip of a fair coin. Any of the six numbers on a fair die is equally likely to come up on a roll. We will refer to a sample space S whose individual elementary events are equally likely as a **uniform sample space**. We then give the following definition of the **probability** of any event in a uniform sample space.

Probability of an Event in a Uniform Sample Space

If S is a finite uniform sample space and E is any event, then the **probability of E**, $P(E)$, is given by

$$P(E) = \frac{\text{Number of elements in } E}{\text{Number of elements in } S} = \frac{n(E)}{n(S)}$$

EXAMPLE 1 Probability for a Single Die Suppose a fair die is rolled and the sample space is $S = \{1,2,3,4,5,6\}$. Determine the probability of each of the following events.
a. The die shows an odd number.
b. The die shows the number 9.
c. The die shows a number less than 8.

Solution **a.** We have $E = \{1,3,5\}$. Then

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

b. The event F that the die shows a 9 is the impossible event. So $n(F) = 0$ and

$$P(F) = \frac{n(F)}{n(S)} = \frac{0}{6} = 0$$

c. The event G that the die shows a number less than 8 is just the certainty event. So $G = \{1,2,3,4,5,6\} = S$ and

$$P(G) = \frac{n(G)}{n(S)} = \frac{6}{6} = 1$$

✦

HISTORICAL NOTE

The Beginnings of Probability

In 1654 the famous mathematician Blaise Pascal had a friend, Chevalier de Mere, a member of the French nobility and a gambler, who wanted to adjust gambling stakes so that he would be assured of winning if he played long enough. This gambler raised questions with Pascal such as the following: In eight throws of a die a player attempts to throw a one, but after three unsuccessful trials the game is interrupted. How should he be compensated? Pascal wrote to a leading mathematician of that day, Pierre de Fermat (1601–1665), about these problems, and their resulting correspondence represents the beginnings of the modern theory of mathematical probability.

EXAMPLE 2 Probability for a Single Card Suppose a single card is randomly drawn from a standard 52-card deck. Determine the probability of each of the following events.
a. A king is drawn.
b. A heart is drawn.

Solution **a.** The event is $E = \{ K\diamondsuit, K\heartsuit, K\spadesuit, K\clubsuit\}$. So,

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}$$

b. The event F contains 13 hearts. So

$$P(F) = \frac{n(F)}{n(S)} = \frac{13}{52} = \frac{1}{4}$$

✦

EXAMPLE 3 Probability for Transistors A bin contains 15 identical (to the eye) transistors except that 6 are defective and 9 are not. What is the probability that a transistor selected at random is defective?

Solution Let us denote the set S to be the set of all 15 transistors and the set E to be the set of defective transistors. Then,

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{15} = \frac{2}{5}$$

✦

REMARK: What if we selected two transistors or two cards? We will learn how to handle this type of experiment in the next chapter.

EXAMPLE 4 Probability for Two Coin Flips A fair coin is flipped twice to observe whether heads or tails shows; order is important. What is the probability that tails occurs both times?

Solution The sample space S consists of the 4 outcomes

$$S = \{(H,H),(H,T),(T,H),(T,T)\}$$

Since we are using a fair coin, each of the individual four elementary events are equally likely. The set E that tails occurs both times is $E = \{(T, T)\}$ and contains one element. We have

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}$$

✦

> ***HISTORICAL NOTE***
>
> **The Beginnings of Empirical Probability**
>
> Empirical probability began with the emergence of insurance companies. Insurance seems to have been originally used to protect merchant vessels and was in use even in Roman times. The first marine insurance companies began in Italy and Holland in the 14th century and spread to other countries by the 16th century. In fact, the famous Lloyd's of London was founded in the late 1600s. The first life insurance seems to have been written in the late 16th century in Europe. All of these companies naturally needed to know with what the likelihood of certain events would occur. The empirical probabilities were determined by collecting data over long periods of time.

✧ Empirical Probability

A very important type of problem that arises every day in business and science is to find a practical way to estimate the likelihood of certain events. For example, a food company may seek a practical method of estimating the likelihood that a new type of candy will be enjoyed by consumers. The most obvious procedure for the company to follow is to randomly select a consumer, have the consumer taste the candy, and then record the result. This should be repeated many times and the final totals tabulated to give the fraction of tested consumers who enjoy the candy. This fraction is then a practical estimate of the likelihood that all consumers will enjoy this candy. We refer to this fraction or number as **empirical probability**.

The London merchant John Graunt (1620–1674) with the publication of *Natural and Political Observations Made upon the Bills of Mortality* in 1662 seems to have been the first person to have gathered data on mortality rates and determined empirical probabilities from them. The data were extremely difficult to obtain. His then-famous London Life Table is reproduced below, showing the number of survivors through certain ages per 100 people.

Age	0	6	16	26	36	46	56	66	76
Survivors	100	64	40	25	16	10	6	3	1

London Life Table

EXAMPLE 5 **Finding Empirical Probability** Using the London Life Table, find the empirical probability of a randomly chosen person living in London in the first half of the 17th century surviving until age 46.

Solution In the London Life Table $N = 100$. If E is the event "survive to age 46," then according to the table the corresponding number is 10. Thus, the empirical probability of people living in London at that time surviving until age 46 was $10/100 = 0.1$. ✦

Consider now a poorly made die purchased at a discount store. Dice are made by drilling holes in the sides and then backfilling. Cheap dice are, of course, not carefully backfilled. So when a lot of holes are made in a face, such as for a side with 6, and they are not carefully backfilled, that side will not be quite as heavy as the others. Thus a 6 will tend to come up more often on the top. Even a die taken from a craps table in Las Vegas, where the dice are of very high quality, will have some tiny imbalance.

EXAMPLE 6 **Finding Empirical Probability** A die with 6 sides numbered from 1 to 6, such as used in the game of craps, is suspected to be somewhat lopsided. A laboratory has tossed this die 1000 times and obtained the results shown in the table. Find the empirical probability that a 2 will occur and the probability that a 6 will occur.

Outcome	1	2	3	4	5	6
Number Observed	161	179	148	177	210	125

Solution The total number observed is 1000. The number observed for the 2 and 6, respectively is 179 and 125. So dividing these numbers by 1000 gives

$$P(2) = {}^{179}/_{1000} = 0.179$$
$$P(6) = {}^{125}/_{1000} = 0.125$$

✦

CONNECTION
Frederick Mosteller and the Dice Experiment

Frederick Mosteller has been president of the American Association for the Advancement of Science, the Institute of Mathematical Statistics, and the American Statistical Association. He once decided that "It would be nice to see if the actual outcome of a real person tossing real dice would match up with the theory." He then engaged Willard H. Longcor to buy some dice, toss them, and keep careful records of the outcomes. Mr. Longcor then tossed the dice on his floor at home so that the dice would bounce on the floor and then up against the wall and then land back on the floor. After doing this several thousand times his wife became troubled by the noise. He then placed a rug on the floor and on the wall, and then proceeded to quietly toss his dice *millions* of times, keeping careful records of the outcomes. In fact, he was so careful and responsible about his task, that he threw away his data on the first 100,000 tosses, since he had a nagging worry that he might have made some mistake keeping perfect track.

✧ Probability Distribution Tables

A **probability distribution table** is a useful way to display probability data for an experiment. In a probability distribution table there is one column (or row) for the events that take place and one column (or row) for the probability of the event. The events chosen must be mutually exclusive and therefore the total probability will add to 1. This is best demonstrated through an example.

EXAMPLE 7 **Flipping a Coin Twice** Write the probability distribution table for the number of heads when a coin is flipped twice.

Solution Recall from Example 4 that the uniform sample space is $S = \{(H,H),(H,T),(T,H),(T,T)\}$. Next, we are asked to organize the events by the number of heads, so we will have three events

$E_1 = \{(H,H)\}$ with two heads and a probability of ${}^1/_4$,
$E_2 = \{(H,T),(T,H)\}$ with exactly one head and a probability of ${}^2/_4$
$E_3 = \{(T,T)\}$ with zero heads and a probability of ${}^1/_4$.

This is shown in the table on the left. ✦

Event	Probability
2 heads	${}^1/_4$
1 head	${}^1/_2$
0 heads	${}^1/_4$

Note how the list of events covered all the possibilities for the number of heads and that the events are all mutually exclusive. You can't have exactly two heads and exactly one head at the same time! Next see that the sum of the probabilities is equal to one. This will always be the case when your probability distribution table is correct.

EXAMPLE 8 **Sum of the Numbers for Two Dice** Two fair dice are rolled. Find the probability distribution table for the sum of the numbers shown uppermost.

Solution Recall the uniform sample space in Example 2 of the last section for rolling two dice. We see the smallest sum is 2 from the roll (1,1) and the largest sum is 12 from the roll (6,6). Count the number of outcomes in each event to find the probability:

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

✦

EXAMPLE 9 **Weight of Oranges** A crate contains oranges and each orange is carefully weighed. It was found that 12 oranges weighed less than 100 grams, 40 oranges weighed 100 grams or more, but less than 150 grams, 60 oranges weighed 150 grams or more, but less than 200 grams, and 8 oranges weighed 200 grams or more. Organize this information in a probability distribution table

Solution The sample space for this experiment was found in the previous section to be $S = \{w | w > 0, w \text{ in grams}\}$. There are four mutually exclusive events described in this sample space and these form the basis of the probability distribution table. A total of $12 + 40 + 60 + 8 = 120$ oranges were weighed. The empirical probability that an orange weighs less than 100 grams will be the ratio $12/120$. The remaining probabilities are found in the same way. This gives the probability distribution table below where w is the weight of an orange in grams.

Event	Probability
$w < 100$	$12/120 = 1/10$
$100 \leq w < 150$	$40/120 = 1/3$
$150 \leq w < 200$	$60/120 = 1/2$
$w \geq 200$	$8/120 = 2/30$

✦

REMARK: Notice that in the probability distribution table above that there were no gaps and no overlap. It is important to be able to translate the statements like "100 grams or more" into an event $100 \leq w$.

Self-Help Exercises 1.4

1. Two tetrahedrons (4 sided), each with equal sides numbered from 1 to 4, are identical except that one is red and the other white. If the two tetrahedrons are tossed and the number on the bottom face of each is observed, what is the sample space for this experiment? Write the probability distribution table for the sum of the numbers on the bottom of the dice.

2. In the past month 72 babies were born at a local hospital. Each baby was carefully measured and it was found that 10 babies were less than 19 inches long, 12 babies were 19 inches or longer but less than 20 inches long, 32 babies were 20 inches or longer but less than 21 inches long. Organize this information in a probability distribution table.

3. An experiment consists of randomly selecting a letter from the word FINITE and observing it. What is the probability of selecting a vowel?

1.4 Exercises

In Exercises 1 through 4, a fair die is tossed. Find the probabilities of the given events.

1. an even number

2. the numbers 4 or 5

3. a number less than 5

4. any number except 2 or 5

In Exercises 5 through 10, a card is drawn randomly from a standard deck of 52 cards. Find the probabilities of the given events.

5. an ace

6. a spade

7. a red card

8. any number between 3 and 5 inclusive

9. any black card between 5 and 7 inclusive

10. a red 8

In Exercises 11 through 14, a basket contains 3 white, 4 yellow, and 5 black transistors. If a transistor is randomly picked, find the probability of each of the given events.

11. white

12. black

13. not yellow

14. black or yellow

15. A somewhat lopsided die is tossed 1000 times with 1 showing on the top face 150 times. What is the empirical probability that a 1 will show?

16. A coin is flipped 10,000 times with heads showing 5050 times. What is the empirical probability that heads will show?

17. The speed of 500 vehicles on a highway with limit of 55 mph was observed, with 400 going between 55 and 65 mph, 60 going less than 55 mph, and 40 going over 65 mph. What is the empirical probability that a vehicle chosen at random on this highway will be going **a.** under 55 mph, **b.** between 55 and 65 mph, **c.** over 65 mph.

18. In a survey of 1000 randomly selected consumers, 50 said they bought brand A cereal, 60 said they bought brand B, and 80 said they bought brand C. What is the empirical probability that a consumer will purchase **a.** brand A cereal, **b.** brand B, **c.** brand C?

19. A large dose of a suspected carcinogen has been given to 500 white rats in a laboratory experiment. During the next year, 280 rats get cancer. What is the empirical probability that a rat chosen randomly from this group of 500 will get cancer?

20. A new brand of sausage is tested on 200 randomly selected customers in grocery stores with 40 saying they like the product, the others saying they do not. What is the empirical probability that a consumer will like this brand of sausage?

21. Over a number of years the grade distribution in a mathematics course was observed to be

A	B	C	D	F
25	35	80	40	20

What is the empirical probability that a randomly selected student taking this course will receive a grade of A? B? C? D? F?

22. A store sells four different brands of VCRs. During the past year the following number of sales of each of the brands were found.

Brand A	Brand B	Brand C	Brand D
20	60	100	70

What is the empirical probability that a randomly selected customer who buys a VCR at this store will pick brand A’? brand B:’ brand C’? brand D?

23. A somewhat lopsided die is tossed 1000 times with the following results. What is the empirical probability that an even number shows?

1	2	3	4	5	6
150	200	140	250	160	100

24. A retail store that sells sneakers notes the following number of sneakers of each size that were sold last year.

7	8	9	10	11	12
20	40	60	30	40	10

What is the empirical probability that a customer buys a pair of sneakers of size 7 or 12?

25. A fair coin is flipped three times, and heads or tails is observed after each flip. What is the probability of the event "at least 2 heads are observed." Refer to the answer in Exercise 3 in Section 4.3.

26. A fair coin is flipped, and it is noted whether heads or tails show. A fair die is tossed, and the number on the top face is noted. What is the probability of the event "heads shows on the coin and an even number on the die." Refer to the answer in Exercise 4 in Section 4.3.

27. A coin is flipped three times. If heads show, one is written down. If tails show, zero is written down. What is the probability of the event "one is observed at least twice." Refer to the answer in Exercise 5 in Section 4.3.

28. Two fair tetrahedrons (4 sided), each with equal sides numbered from 1 to 4, are identical except that one is red and the other white. If the two tetrahedrons are tossed and the number on the bottom face of each is observed, what is the probability of the event "the sum of the numbers is 4." Refer to the answer in Exercise 6 in Section 4.3.

In Exercises 29 through 34, assume that all elementary events in the same sample space are equally likely.

29. A fair coin is flipped three times. What is the probability of obtaining exactly 2 heads? At least 1 head?

30. A family has three children. Assuming a boy is as likely as a girl to have been born, what is the probability that two are boys and one is a girl? That at least one is a boy?

31. A fair coin is flipped and a fair die is tossed. What is the probability of obtaining a head and a 3?

32. A fair coin is flipped twice and a fair die is tossed. What is the probability of obtaining 2 heads and a 3?

33. A pair of fair dice are tossed. What is the probability of obtaining a sum of 2? 4? 8?

34. A pair of fair dice are tossed. What is the probability of obtaining a sum of 5? 6? ll?

35. An experiment consists of selecting a digit from the number 112964333 and observing it. What is the probability that "an even digit is selected."

36. An experiment consists of selecting a letter from the word CONNECTICUT and observing it. What is the probability that "a vowel is selected."

Solutions to Self-Help Exercises 1.4

1. The sample space was found in the previous section and is

$$\begin{aligned} S = \{ & (1,1),(1,2),(1,3),(1,4),\\ & (2,1),(2,2),(2,3),(2,4),\\ & (3,1),(3,2),(3,3),(3,4),\\ & (4,1),(4,2),(4,3),(4,4)\} \end{aligned}$$

The sum of the numbers ranges from $1+1=2$ to $4+4=8$. Count the outcomes in each event to find

Sum	2	3	4	5	6	7	8
Probability	1/16	2/16	3/16	4/16	3/16	2/16	1/16

2. The sample space for this experiment is $S = \{x | x > 0, x \text{ in inches}\}$. The lengths of $10+12+32=54$ babies is given. A careful examination of the events shows that no mention was made of babies longer than 21 inches. We deduce that $72-54=18$ babies must be 21 inches or longer. This can now be arranged in a probability distribution table where x is the length of the baby in inches.

Event	Probability
$x < 19$	$10/72 = 5/36$
$19 \le x < 20$	$12/72 = 1/6$
$20 \le x < 21$	$32/72 = 4/9$
$x \ge 21$	$18/72 = 1/4$

3. FINITE has six letters and there are three vowels. So

$$P(\text{vowel}) = \frac{3}{6} = \frac{1}{2}$$

1.5 Rules for Probability

✧ Elementary Rules

Recall that if S is a finite uniform sample space, that is, a space for which all individual elementary elements are equally likely, and E is any event, then the probability of E, denoted by $P(E)$, is given by

$$P(E) = \frac{\text{Number of elements in } E}{\text{Number of elements in } S} = \frac{n(E)}{n(S)}$$

If E is an event in a sample space then $0 \le n(E) \le n(S)$. Dividing this by $n(S)$ then gives

$$0 \le \frac{n(E)}{n(S)} \le \frac{n(S)}{n(S)} = 1$$

Using the definition of probability given above yields

$$0 \le P(E) \le 1$$

This is our first rule for probability. Notice also that

$$P(S) = \frac{n(S)}{n(S)} = 1 \quad \text{and} \quad P(\emptyset) = \frac{n(\emptyset)}{n(S)} = \frac{0}{n(S)} = 0$$

These rules apply for events in any sample space; however, the derivations just given are valid only for spaces with equally likely events.

Elementary Rules for Probability

For any event E in a sample space S we have

$$0 \le P(E) \le 1 \qquad P(S) = 1 \qquad P(\emptyset) = 0$$

> ***HISTORICAL NOTE***
>
> **Some Developments in Probability**
>
> Neither Pascal nor Fermat published their initial findings on probability. Christian Huygens (1629–1695) became acquainted with the work of Pascal and Fermat and subsequently published in 1657 the first tract on probability: *On Reasoning in Games of Dice*. This little pamphlet remained the only published work on probability for the next 50 years. James Bernoulli (1654–1705) published the first substantial tract on probability when his *Art of Reasoning* appeared 7 years after his death. This expanded considerably on Huygens' work. The next major milestone in probability occurred with the publication in 1718 of Abraham De Moivre's work *Doctrine of Chance: A Method of Calculating of Events in Play*. Before 1770, probability was almost entirely restricted to the study of gambling and actuarial problems, although some applications in errors of observation, population, and certain political and social phenomena had been touched on. It was Pierre Simon Laplace (1739–1827) who broadened the mathematical treatment of probability beyond games of chance to many areas of scientific research. The theory of probability undoubtedly owes more to Laplace than to any other individual.

✧ Union Rule for Probability

We would now like to determine the probability of the union of two events E and F. We start by recalling the union rule for sets:

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

Now divide both sides of this last equation by $n(S)$ and obtain

$$\frac{n(E \cup F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)}$$

By the definition of probability given above this becomes

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

This is called the **union rule for probability**. This rule applies for events in any sample space; however, the derivation just given is valid only for spaces with equally likely events.

> **Union Rule for Probability**
>
> $$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

EXAMPLE 1 **Union Rule With Drawing a Card** A single card is randomly drawn from a standard deck of cards. What is the probability that it will be a red card or a king.

Solution Let R be the set of red cards and let K be the set of kings. Red cards consist of hearts and diamonds, so there are 26 red cards. Therefore $P(R) = 26/52$. There are 4 kings, so $P(K) = 4/52$. Among the 4 kings, there are 2 red cards. So, $P(R \cap K) = 2/52$. Using the union rule gives

$$\begin{aligned} P(R \cup K) &= P(R) + P(K) - P(R \cap K) \\ &= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} \\ &= \frac{28}{52} = \frac{7}{13} \end{aligned}$$

✦

REMARK: It is likely that you would intuitively use the union rule had you been asked to pick out all of the cards from the deck that were red or kings. You would choose out all of the red cards along with all of the kings for a total of 28 cards.

EXAMPLE 2 **Union Rule With Two Dice** Two dice, identical except that one is green and the other is red, are tossed and the number of dots on the top face of each is observed. Let E consist of those outcomes for which the number of dots on the top face of the green dice is a 1 or 2. Let F be the event that the sum of the number of dots on the top faces of the two dice is 6. Find the probability that a 1 or 2 will be on the top of the green die or the sum of the two numbers will be 6.

E	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
F	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Figure 1.21

Solution Notice that

$$\begin{aligned} E &= \{(1,1),(1,2),\cdots,(1,6),(2,1),(2,2),\cdots,(2,6)\} \\ F &= \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \\ E \cap F &= \{(1,5),(2,4)\} \end{aligned}$$

The set that a 1 or 2 will be on the top of the green die and the sum of the two numbers will be 6 is $E \cap F$. To find $p(E \cap F)$ use the union rule of probability and obtain

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) - P(E \cap F) \\ &= \frac{12}{36} + \frac{5}{36} - \frac{2}{36} = \frac{15}{36} \end{aligned}$$

Alternatively, you can draw the sample space for two dice and circle all outcomes that have a 1 or 2 on the top of the green die or the sum of the two numbers shown uppermost is 6. This is done in Figure 1.21. Counting the circled outcomes we find there are 15 of them. ✦

Consider two events E and F that are mutually exclusive, that is, $E \cap F) = \emptyset$. Then $P(E \cap F) = 0$. Using the union rule of probability for these two sets gives

$$\begin{aligned} P(E \cup F) &= P(E) + P(F) - P(E \cap F) \\ &= P(E) + P(F) - 0 \\ &= P(E) + P(F) \end{aligned}$$

We then have the following rule:

Union Rule for Mutually Exclusive Events

If E and F are mutually exclusive events, then

$$P(E \cup F) = P(E) + P(F)$$

For any event E in a sample space, $E \cup E^c = S$ and $E \cap E^c = \emptyset$. So, E and E^c are mutually exclusive. Using the union rule for mutually exclusive events we have that

$$P(E) + P(E^c) = P(E \cup E^c) = P(S) = 1$$

So, $P(E^c) = 1 - P(E)$ and $P(E) = 1 - P(E^c)$. We call this the **complement rule**.

Complement Rule for Probability

$$P(E^c) = 1 - P(E) \qquad P(E) = 1 - P(E^c)$$

EXAMPLE 3 **Complement Rule for Two Dice** Consider the dice described in Example 2. What is the probability that the sum of the two numbers is less than 12.

Solution Let E be the event that the sum of the two numbers is less than 12. Then we wish to find $P(E)$. It is tedious to find this directly. Notice that $E^c = \{(6,6)\}$. Now use the complement rule.

$$P(E) = 1 - P(E^c) = 1 - \frac{1}{36} = \frac{35}{36}$$ ✦

EXAMPLE 4 Finding Empirical Probability A die with 6 sides numbered from 1 to 6, such as used in the game of craps, is suspected to be somewhat lopsided. A laboratory has tossed this die 1000 times and obtained the results shown in the table. Find the empirical probability that an even number will occur.

Outcome	1	2	3	4	5	6
Number Observed	161	179	148	177	210	125

Solution The total number observed is 1000. The number observed for the 2, 4, and 6, respectively is 179, 177, and 125. So dividing these numbers by 1000 gives

$$P(2) = 179/1000 = 0.179$$
$$P(4) = 177/1000 = 0.177$$
$$P(6) = 125/1000 = 0.125$$

To find the empirical probability of an even number these three values can be added as the events are mutually exclusive. That is,

$$P(\text{even}) = P(2) + P(4) + P(6) = 0.179 + 0.177 + 0.125 = 0.481$$ ✦

EXAMPLE 5 Finding the Probability of an Event A salesman makes two stops when in Pittsburgh. The first stop yields a sale 10% of the time, the second stop 15% of the time, and both stops yield a sale 4% of the time. What proportion of the time does a trip to Pittsburgh result in no sales?

Solution Let E be the event a sale is made at the first stop and F the event that a sale is made at the second stop. What should we make of the statement that the first stop yields a sale 10% of the time. It seems reasonable to assume that the salesman or his manager have looked at his sales data and estimated the 10% number. We then take the 10% or 0.10 as the empirical probability. We interpret the other percentages in a similar way. We then have

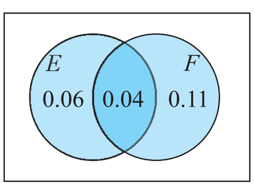


Figure 1.22

$$P(E) = 0.10 \qquad P(F) = 0.15 \qquad P(E \cap F) = 0.04$$

Since $P(E \cap F) = 0.04$, we place 0.04 in the region $E \cap F$ in Figure 1.22. Now since $P(E) = 0.10$, we can see that $P(E \cap F^c) = 0.10 - 0.04 = 0.06$. In a similar fashion we have $P(E^c \cap F) = 0.15 - 0.04 = 0.11$. Thus, we readily see from Figure 1.22 that

$$P(E \cup F) = 0.06 + 0.04 + 0.11 = 0.21$$

Then by the complement rule we have

$$P((E \cup F)^c) = 1 - P(E \cup F) = 1 - 0.21 = 0.79$$

Thus no sale is made in Pittsburgh 79% of the time. ✦

REMARK: We could have obtained $P(E \cup F)$ directly from the union rule as follows:

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.10 + 0.15 - 0.04 = 0.21$$

EXAMPLE 6 Finding the Probability of an Event The probability that any of the first five numbers of a loaded die will come up is the same while the probability that a 6 comes up is 0.25. What is the probability that a 1 will come up?

Solution We are given $P(1) = P(2) = P(3) = P(4) = P(5)$, $P(6) = 0.25$. Also, all the probabilities must add up to 1, so

$$\begin{aligned} 1 &= P(1) + P(2) + P(3) + P(4) + P(5) + P(6) \\ &= 5P(1) + 0.25 \\ 5P(1) &= 0.75 \\ P(1) &= 0.15 \end{aligned}$$

✦

EXAMPLE 7 Continuous Sample Space Arrange the following information in a probability distribution table: A crop of apples is brought in for weighing. It is found that 10% of the apples weigh less than 100 gm, 40% weigh 200 gm or less, and 25% weigh more than 300 gm.

Solution If we let x = weight of an apple in grams then

Event	Probability
$0 \leq x < 100$ gm	0.10
100 gm $\leq x \leq$ 200 gm	0.30
200 gm $< x \leq$ 300 gm	0.35
$x >$ 300 gm	0.25

Note that the 40% of the apples that weigh 200 gm or less includes the 10% that weigh less than 100 grams. Since the events in a probability distribution table must be mutually exclusive, the 30% that weigh 100 grams or more and 200 grams or less are shown in the second row. The third row of the table is found using deductive reasoning as the total probability must be 1 and there is a gap in the events. ✦

✧ Odds (Optional)

One can interpret probabilities in terms of **odds** in a bet. Suppose in a sample space S we are given an event E with probability $P = P(E) = \frac{5}{7}$. In the long term we expect E to occur 5 out of 7 times. Now, $P(E^c) = \frac{2}{7}$ and in the long term we expect that E^c to occur 2 out of 7 times. Then we say that the **odds in favor** of E are 5 to 2.

Odds

The **odds in favor** of an event E are defined to be the ratio of $P(E)$ to $P(E^c)$, or

$$\frac{P(E)}{P(E^c)} = \frac{P(E)}{1-P(E)}$$

Often the ratio ${P(E)}/{P(E^c)}$ is reduced to lowest terms, a/b, and then we say that **the odds are a to b** or a:b.

EXAMPLE 8 **Determining the Odds of an Event** You believe that a horse has a probability of $1/4$ of winning a race. What are the odds of this horse winning? What are the odds of this horse losing? What profit should a winning \$2 bet return to be fair?

Solution Since the probability of winning is $P = 1/4$, the odds of winning are

$$\frac{P}{1-P} = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{3}$$

that is: 1 to 3 or 1:3.

Since the probability of winning is $\frac{1}{4}$, the probability of losing is $1-1/4 = 3/4$. Then the odds for losing is

$$\frac{3/4}{1-3/4} = \frac{3/4}{1/4} = \frac{3}{1}$$

or 3 to 1 or 3:1. Since the fraction $3/1$ can also be written as $6/2$ with odds 6 to 2, a fair \$2 bet should return \$6 for a winning ticket. ✦

Notice that making this same bet many times, we expect to win \$6 one-fourth of the time and lose \$2 three-fourths of the time. So, for example, on every four bets we would expect to win \$6 once and lose \$2 three times. Our average winnings would be $6(1)-2(3) = 0$ dollars.

If the odds for an event E are given as a/b, we can calculate the probability $P(E)$. We have

$$\begin{aligned} \frac{a}{b} &= \frac{P(E)}{1-P(E)} \\ a(1-P(E)) &= bP(E) \\ a &= bP(E)+aP(E) \\ &= P(E)(a+b) \\ P(E) &= \frac{a}{a+b} \end{aligned}$$

Obtaining Probability From Odds

Suppose that the odds for an event E occurring is given as a/b or $a:b$, then

$$P(E) = \frac{a}{a+b}$$

REMARK: One can think of the odds a:b of event E as saying if this experiment was carried out $a+b$ times, then a of those times E would have occurred. Our definition of empirical probability then says $P(E) = \frac{a}{a+b}$

EXAMPLE 9 **Obtaining Probability From Odds** At the race track, the odds for a horse winning is listed at $3/2$. What is the probability that the horse will win.

Solution Using the above formula for odds a/b, we have

$$P = \frac{a}{a+b} = \frac{3}{3+2} = 0.60$$ ✦

Self-Help Exercises 1.5

1. If $S = \{a, b, c\}$ with $P(a) = P(b) = 2P(c)$, find $P(a)$.
2. A company has bids on two contracts. They believe that the probability of obtaining the first contract is 0.4 and of obtaining the second contract is 0.3, while the probability of obtaining both contracts is 0.1.
   a. Find the probability that they will obtain exactly one of the contracts.
   b. Find the probability that they will obtain neither of the contracts.
3. What are the odds that the company in the previous exercise will obtain both of the contracts?

1.5 Exercises

In all the following, S is assumed to be a sample space.

1. Let $S = \{a, b, c\}$ with $P(a) = 0.1$, $P(b) = 0.4$, and $P(c) = 0.5$. Let $E = \{a, b\}$ and $F = \{b, c\}$. Find $P(E)$ and $P(F)$.
2. Let $S = \{a, b, c, d, e, f\}$ with $P(a) = 0.1$, $P(b) = 0.2$, $P(c) = 0.25$, $P(d) = 0.15$, $P(e) = 0.12$, and $P(f) = 0.18$. Let $E = \{a, b, c\}$ and $F = \{c, d, e, f\}$ and find $P(E)$ and $P(F)$.
3. Let $S = \{a, b, c, d, e, f\}$ with $P(b) = 0.2$, $P(c) = 0.25$, $P(d) = 0.15$, $P(e) = 0.12$, and $P(f) = 0.1$. Let $E = \{a, b, c\}$ and $F = \{c, d, e, f\}$. Find $P(a)$, $P(E)$, and $P(F)$.
4. Let $S = \{a, b, c, d, e, f\}$ with $P(b) = 0.3$, $P(c) = 0.15$, $P(d) = 0.05$, $P(e) = 0.2$, $P(f) = 0.13$. Let $E = \{a, b, c\}$ and $F = \{c, d, e, f\}$. Find $P(a)$, $P(E)$, and $P(F)$.
5. If $S = \{a, b, c, d\}$ with $P(a) = P(b) = P(c) = P(d)$, find $P(a)$.
6. If $S = \{a, b, c\}$ with $P(a) = P(b)$ and $P(c) = 0.4$, find $P(a)$.
7. If $S = \{a, b, c, d, e, f\}$ with $P(a) = P(b) = P(c) = P(d) = P(e) = P(f)$, find $P(a)$.
8. If $S = \{a, b, c\}$ with $P(a) = 2P(b) = 3P(c)$, find $P(a)$.
9. If $S = \{a, b, c, d, e, f\}$ with $P(a) = P(b) = P(c)$, $P(d) = P(e) = P(f) = 0.1$, find $P(a)$.
10. If $S = \{a, b, c, d, e, f\}$ and if $P(a) = P(b) = P(c)$, $P(d) = P(e) = P(f)$, $P(d) = 2P(a)$, find $P(a)$.
11. If E and F are two disjoint events in S with $P(E) = 0.2$ and $P(F) = 0.4$, find $P(E \cup F)$, $P(E^c)$, and $P(E \cap F)$.
12. Why is it not possible for E and F to be two disjoint events in S with $P(E) = 0.5$ and $P(F) = 0.7$?
13. If E and F are two disjoint events in S with $P(E) = 0.4$ and $P(F) = 0.3$, find $P(E \cup F)$, $P(F^c)$, $P(E \cap F)$, $P((E \cup F)^c)$, and $P((E \cap F)^c)$.
14. Why is it not possible for $S = \{a, b, c\}$ with $P(a) = 0.3$, $P(b) = 0.4$, and $P(c) = 0.5$?

15. Let E and F be two events in S with $P(E) = 0.3$, $P(F) = 0.5$, and $P(E \cap F) = 0.2$. Find $P(E \cap F)$ and $P(E \cap F^c)$.

16. Let E and F be two events in S with $P(E) = 0.3$, $P(F) = 0.5$, and $P(E \cap F) = 0.2$. Find $P(E^c \cap F)$ and $P(E^c \cap F^c)$.

17. Let E and F be two events in S with $P(E) = 0.3$, $P(F) = 0.5$, and $P(E \cup F) = 0.6$. Find $P(E \cap F)$ and $P(E \cap F^c)$.

18. Why is it not possible to have E and F two events in S with $P(E) = 0.3$ and $P(E \cap F) = 0.5$?

In Exercises 19 through 22, let E, F, and G be events in S with $P(E) = 0.55$, $P(F) = 0.4$, $P(G) = 0.45$, $P(E \cap F) = 0.3$, $P(E \cap G) = 0.2$, $P(F \cap G) = 0.15$, and $P(E \cap F \cap G) = 0.1$.

19. Find $P(E \cap F \cap G^c)$, $P(E \cap F^c \cap G)$, and $P(E \cap F^c \cap G^c)$.

20. Using the results of the previous exercise, find $P(E^c \cap F \cap G)$, $P(E^c \cap F \cap G^c)$, and $P(E^c \cap F^c \cap G)$.

21. Using the results of the previous two exercises, find $P(E \cup F \cup G)$.

22. Using the results of the previous three exercises, find $P(E^c \cup F^c \cup G^c)$.

23. For the loaded die in Example 6 of the text, what are the odds that

a. a 2 will occur **b.** a 6 will occur?

24. For the loaded die in Example 6 of the text, what are the odds that

a. a 3 will occur **b.** a 1 will occur?

25. A company believes it has a probability of 0.40 of receiving a contract. What is the odds that it will?

26. In Example 5 of the text, what are the odds that the salesman will make a sale on

a. the first stop **b.** on the second stop

c. on both stops?

27. It is known that the odds that E will occur are 1:3 and that the odds that F will occur are 1:2, and that both E and F cannot occur. What are the odds that E or F will occur?

28. If the odds for a successful marriage are 1:2, what is the probability for a successful marriage?

29. If the odds for the Giants winning the World Series are 1:4, what is the probability that the Giants will win the Series?

Applications

30. Bidding on Contracts An aerospace firm has three bids on government contracts and knows that the contracts are most likely to be divided up among a number of companies. The firm decides that the probability of obtaining exactly one contract is 0.6, of exactly two contracts is 0.15, and of exactly three contracts is 0.04. What is the probability that the firm will obtain at least one contracts? No contracts?

31. Quality Control An inspection of computers manufactured at a plant reveals that 2% of the monitors are defective, 3% of the keyboards are defective, and 1% of the computers have both defects.

a. Find the probability that a computer at this plant has at least one of these defects.

b. Find the probability that a computer at this plant has none of these defects.

32. Medicine A new medication produces headaches in 5% of the users, upset stomach in 15%, and both in 2%.

a. Find the probability that at least one of these side effects occurs.

b. Find the probability that neither of these side effects occurs.

33. Manufacturing A manufactured item is guaranteed for one year and has three critical parts. It has been decided that during the first year the probability of failure of the first part is 0.03, of the second part 0.02, the third part 0.01, both the first and second is 0.005, both the first and third is 0.004, both the second and third is 0.003, and all three parts 0.001.

a. What is the probability that exactly one of these parts will fail in the first year?

b. What is the probability that at least one of these parts will fail in the first year?

c. What is the probability that none of these parts will fail in the first year?

34. Marketing A survey of business executives found that 40% read *Business Week*, 50% read *Fortune*, 40% read *Money*, 17% read both *Business Week* and *Fortune*, 15% read both both *Business Week* and *Money*, 14% read both *Fortune* and *Money*, and 8% read all three of these magazines.

a. What is the probability that one of these executives reads exactly one of these three magazines?

b. What is the probability that one of these executives reads at least one of these three magazines?

c. What is the probability that one of these executives reads none of these three magazines?

35. Advertising A firm advertises three different products, A, B, and C, on television. From past experience, it expects 1.5% of listeners to buy exactly one of the products, 1% to buy exactly two of the products, 1.2% to buy A, 0.4% to buy both A and B, 0.3% to buy both A and C, and 0.6% to buy A but not the other two.

a. Find the probability that a listener will buy only B or only C.

b. Find the probability that a listener will buy all three.

c. Find the probability that a listener will buy both B and C.

d. Find the probability that a listener will buy none of the three.

36. Sales A salesman always makes a sale at one of the three stops in Atlanta and 30% of the time makes a sale at only the first stop, 15% at only the second stop, 20% at only the third stop, and 35% of the time at exactly two of the stops. Find the probability that the salesman makes a sale at all three stops in Atlanta.

Extensions

37. Let E and F be two events in S with $P(E) = 0.5$ and $P(F) = 0.7$. Just how small could $P(E \cap F)$ possibly be?

38. Let E and F be two events in S with $P(E) = 0.3$ and $P(F) = 0.4$. Just how large could $P(F \cup F)$ possibly be?

39. You buy a new die and toss it 1000 times. A 1 comes up 165 times. Is it true that the probability of a 1 showing on this die is 0.165?

40. A fair coin is to be tossed 100 times. Naturally you expect tails to come up 50 times. After 60 tosses, heads has come up 40 times. Is it true that now heads is likely to come up less often than tails during the next 40 tosses?

41. You are playing a game at a casino and have correctly calculated the probability of winning any one game to be 0.48. You have played for some time and have won 60% of the time. You are on a roll. Should you keep playing?

42. You are watching a roulette game at a casino and notice that red has been coming up unusually often. (Red should come up as often as black.) Is it true that, according to "the law of averages," black is likely to come up unusually often in the next number of games to "even things up"?

43. You buy a die and toss it 1000 times and notice that a 1 came up 165 times. You decide that the probability of a 1 on this die is 0.165. Your friend takes this die and tosses it 1000 times and notes that a 1 came up 170 times. He concludes that the probability of a 1 is 0.17. Who is correct?

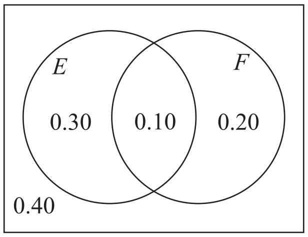
44. People who frequent casinos and play lotteries are gamblers, but those who run the casinos and lotteries are not. Do you agree? Why or why not?

Solutions to Self-Help Exercises 1.5

1. If $S = \{a, b, c\}$ and $P(a) = P(b) = 2P(c)$, then

$$\begin{aligned} 1 &= P(a) + P(b) + P(c) \\ &= P(a) + P(a) + 0.5P(a) \\ &= 2.5P(a) \\ P(a) &= 0.4 \end{aligned}$$

2. a. Let E be the event that the company obtains the first contract and let F be the event that the company obtains the second contract. The event that the company obtains the first contract but not the second is $E \cap F^c$, while



the event that the company obtains the second contract but not the first is $E^c \cap F$. These two sets are mutually exclusive, so the probability that the company receives exactly one of the contracts is

$$P(E \cap F^c) + P(E^c \cap F)$$

Now $P(E) = 0.40$, $P(F) = 0.30$, and since $E \cap F$ is the event that the company receives both contracts, $P(E \cap F) = 0.10$. Notice on the accompanying diagram that $E \cap F^c$ and $E \cap F$ are mutually disjoint and that $(E \cap F^c) \cup (E \cap F) = E$. Thus

$$\begin{aligned} P(E \cap F^c) + P(E \cap F) &= P(E) \\ P(E \cap F^c) + 0.10 &= 0.40 \\ P(E \cap F^c) &= 0.30 \end{aligned}$$

Also notice on the accompanying diagram that $E^c \cap F$ and $E \cap F$ are mutually disjoint and that $(E^c \cap F) \cup (E \cap F) = F$. Thus

$$\begin{aligned} P(E^c \cap F) + P(E \cap F) &= P(F) \\ P(E^c \cap F) + 0.10 &= 0.30 \\ P(E \cap F^c) &= 0.20 \end{aligned}$$

Thus the probability that the company will receive exactly one of the contracts is

$$P(E \cap F^c) + P(E^c \cap F) = 0.30 + 0.20 = 0.50$$

b. The event that the company obtains neither contract is given by $(E \cup F)^c$. From the diagram

$$P(E \cup F) = 0.30 + 0.10 + 0.20 = 0.60$$

Thus

$$P((E \cup F)^c) = 1 - P(E \cup F) = 1 - 0.60 = 0.40$$

The probability that the company receives neither contract is 0.40.

1.6 Conditional Probability

APPLICATION
Locating Defective Parts

A company makes the components for a product at a central location. These components are shipped to three plants, 1 , 2, and 3, for assembly into a final product. The percentages of the product assembled by the three plants are, respectively, 50%, 20%, and 30%. The percentages of defective products coming from these three plants are, respectively, 1%, 2%, and 3%. What is the probability of randomly choosing a product made by this company that is defective from Plant 1? See Example 4 for the answer.

✧ Definition of Conditional Probability

The probability of an event is often affected by the occurrences of other events. For example, there is a certain probability that an individual will die of lung cancer. But if a person smokes heavily, then the probability that this person will die of lung cancer is higher. That is, the probability has changed with the additional information. In this section we study such conditional probabilities. This important idea is further developed in the next section.

Given two events E and F, we call the probability that E will occur given that F has occurred the **conditional probability** and write $P(E|F)$. Read this as "*the probability of E given F.*"

Conditional probability normally arises in situations where the old probability is "updated" based on new information. This new information is usually a change in the sample space. Suppose, for example, a new family moves in next door. The real estate agent mentions that this new family has two children. Based on this information, you can calculate the probability that both children are boys. Now a neighbor mentions that they met one child from the new family and this child is a boy. Now the probability that both children from the new family are boys has changed given this new information. We will find this new probability in Example 2.

EXAMPLE 1 **Finding Conditional Probability** A card is drawn randomly from a deck of 52 cards.

a. What is the probability that this card is an ace?

b. What is the probability that this card is an ace given that the card is known to be red and 10 or higher?

Solution

a. The uniform sample space consists of 52 cards and has the uniform probability. Thus, if $E = \{x|x \text{ is an ace}\}$,

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52}$$

.

b. Let F be the event in S consisting of all red cards 10 or higher. We have $n(S) = 52$ and $n(F) = 10$. Since we are certain that the card chosen was red and 10 or higher, our sample space is no longer the entire deck of 52 cards, but simply those 10 cards in F. This will be our denominator in the ratio of outcome in our event to outcomes in our sample space. The event ace and known to be red and 10 or higher, has two outcomes as there are two red aces. So the numerator will be $n(E \cap F) = 2$. Putting this together we have the probability of E given that F has occurred is

$$P(E|F) = \frac{n(E \cap F)}{n(F)} = \frac{2}{10}$$ ✦

It can be helpful to divide the numerator and denominator in the last fraction by $n(S)$ and obtain

$$P(E|F) = \frac{n(E \cap F)}{n(F)} = \frac{n(E \cap F)/n(S)}{n(F)/n(S)}$$

But this is just

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

This motivates the following definition for any event E and F in a sample space S.

Conditional Probability
Let E and F be two events in a sample space S. The **conditional probability** that E occurs given that F has occurred is defined to be

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

REMARK: It is worthwhile to notice that the events E, F, and $E \cap F$ are all in the original sample space S and $P(E)$, $P(F)$, and $P(E \cap F)$ are all probabilities defined on S. However, we can think of $P(E|F)$ as a probability defined on the new sample space $S' = F$. See Figure 1.23.

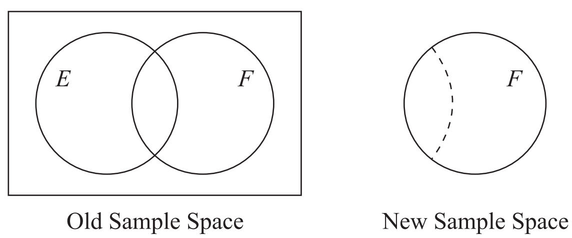


Figure 1.23

EXAMPLE 2 **Calculating a Conditional Probability** A new family has moved in next door and is known to have two children. Find the probability that both children are boys given that at least one is a boy. Assume that a boy is as likely as a girl.

Solution Let E be the event that both children are a boy and F the event that at least one is a boy. Then

$$S = \{BB, BG, GB, GG\}, E = \{BB\}, F = \{BB, BG, GB\}, E \cap F = \{BB\}$$

and

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}$$ ✦

✧ The Product Rule

We will now see how to write $P(E \cap F)$ in terms of a product of two probabilities. From the definition of conditional probability, we have

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, \qquad P(F|E) = \frac{P(F \cap E)}{P(E)}$$

if $P(E) > 0$ and $P(F) > 0$. Solving for $P(E \cap F)$ and $P(F \cap E)$, we obtain

$$P(E \cap F) = P(F)P(E|F), \qquad P(F \cap E) = P(E)P(F|E)$$

Since $P(E \cap F) = P(F \cap E)$, it follows that

$$P(E \cap F) = P(F)P(E|F) = P(E)P(F|E)$$

This is called the product rule.

Product Rule
If E and F are two events in a sample space S with $P(E) > 0$ and $P(F) > 0$, then

$$P(E \cap F) = P(F)P(E|F) = P(E)P(F|E)$$

EXAMPLE 3 **Using the Product Rule** Two bins contain transistors. The first bin has 5 defective and 15 non-defective transistors while the second bin has 3 defective and 17 non-defective transistors. If the probability of picking either bin is the same, what is the probability of picking the first bin and a good transistor?

Solution The sample space is $S = \{1D, 1N, 2D, 2N\}$ where the number refers to picking the first or second bin and the letter refers to picking a defective (D) or non-defective (N) transistor.

If E is the event "pick the first bin" and F is the event "pick a non-defective transistor," then $E = \{1D, 1N\}$ and $F = \{1N, 2N\}$. The probability of picking a non-defective transistor given that the first bin has been picked is the conditional probability $P(F|E) = 15/20$. The event "picking the first bin and a non-defective transistor" is $E \cap F$. From the product rule

$$P(E \cap F) = P(E)P(F|E) = \frac{1}{2} \cdot \frac{15}{20} = \frac{3}{8}$$ ✦

✧ Probability Trees

We shall now consider a finite sequence of experiments in which the outcomes and associated probabilities of each experiment depend on the outcomes of the preceding experiments. For example, we can choose a card from a deck of cards, place the picked card on the table, and then pick another card from the deck. This process could continue until all the cards are picked.

Such a finite sequence of experiments is called a **finite stochastic process**. Stochastic processes can be effectively described by **probability trees** that we now consider. The following example should be studied carefully since we will return to it in the next section.

EXAMPLE 4 **Using a Probability Tree** A company makes the components for a product at a central location. These components are shipped to three plants, 1, 2, and 3, for assembly into a final product. The percentage of the product assembled by the three plants are, respectively, 50%, 20%, and 30%. The percentages of defective products coming from these three plants are, respectively, 1%, 2%, and 3%.

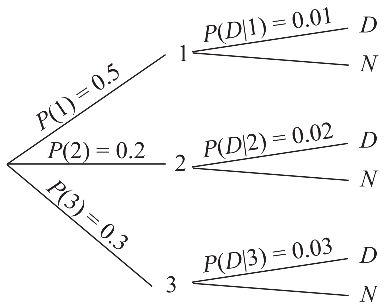


Figure 1.24

a. What is the probability of randomly choosing a product made by this company that is defective from Plant 1?, 2?, 3?

b. What is the probability of randomly choosing a product made by this company that is defective?

Solution The probability of the part being assembled in Plant 1 is $P(1) = 0.5$, in Plant 2 is $P(2) = 0.2$, and in Plant 3 is $P(3) = 0.3$. We begin our tree with this information as shown in Figure 1.24. Notice how this is similar to the trees drawn in earlier sections with the only difference being the probability of the outcome being placed on the branch leading to that outcome.

We are also given the conditional probabilities, $P(D|1) = 0.01$, $P(D|2) = 0.02$, and $P(D|3) = 0.03$. The branch leading from Plant 1 to a defective item tells us that we are referring to items from Plant 1 and therefore the number that should be placed on this branch is the probability the component is defective **given** that it was made at Plant 1, $P(D|1) = 0.01$. Continue with the conditional probabilities given for the other plants. This is all shown in the tree diagram in Figure 1.24.

a. Notice that the product rule, $P(1 \cap D) = P(1)P(D|1)$, represents multiplying along the branches 1 and D. So using the product rule we have

$$
\begin{aligned}
P(1 \cap D) &= P(1)P(D|1) = (0.5)(0.01) = 0.005 = 0.5\% \\
P(2 \cap D) &= P(2)P(D|2) = (0.2)(0.02) = 0.004 = 0.4\% \\
P(3 \cap D) &= P(3)P(D|3) = (0.3)(0.03) = 0.009 = 0.9\%
\end{aligned}
$$

b. For a component to be defective it came from Plant 1 or Plant 2 or Plant 3. These events are mutually exclusive since a component can only come from one plant. We have the following:

$$
\begin{aligned}
P(D) &= P(1 \cap D) + P(2 \cap D) + P(3 \cap D) \\
&= 0.005 + 0.004 + 0.009 = 0.018
\end{aligned}
$$

✦

EXAMPLE 5 **Using a Probability Tree** A box contains three blue marbles and four red marbles. A marble is selected at random until a red one is picked.

a. What is the probability that the number of marbles selected is one?

b. What is the probability that the number of marbles selected is two?

c. What is the probability that the number of marbles selected is three?

Solution The process for selecting a marble one at a time from the box is shown in Figure 1.25.

a. On the first draw there are four red marbles in a box of seven. So the probability of selecting a red marble is $4/7$. We note for further reference that the probability of selecting a blue marble is $3/7$. These probabilities are on the first legs of the tree diagram.

b. The only way it can take two selections to obtain a blue marble is for the first selection be a blue marble. In this case there are two blue marbles and four red marbles. Thus, the probability of selecting a red marble given that a blue marble was chosen first is $4/6$.

From the tree in Figure 1.25, the probability of the branch B_1R_2 is $(3/7) \cdot (4/6) = 2/7$. We note for further reference that the probability of selecting a blue marble on the second selection given that a blue marble was selected first is $2/6$. See Figure 1.25.

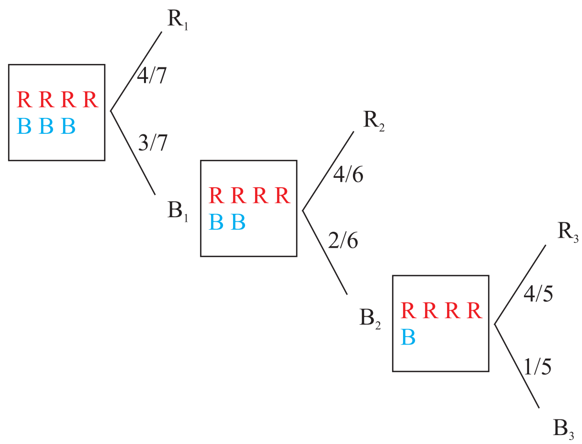


Figure 1.25

c. The only way it can take three selections to obtain a red marble is for the first two selections to be blue marbles. At this point (with two blue marbles missing) there is one blue marble and four red marbles left in the box. The probability of selecting a red marble given that a blue marble was chosen the first and second time is $4/5$. Now the probability of the branch $B_1B_2R_3$ can be found by applying the product rule (multiply along the branches) to find

$$\frac{3}{7} \cdot \frac{2}{6} \cdot \frac{4}{5} = \frac{4}{35}$$ ✦

✧ Independent Events

We say that two events E and F are **independent** if the outcome of one does not affect the outcome of the other. For example, the probability of obtaining a head on a second flip of a coin is independent from what happened on the first flip. This is intuitively clear since the coin cannot have any memory of what happened on the first flip. Indeed the laws of physics determine the probability of heads occurring. Thus for the probability of heads to be different on the second flip the laws of physics must be different on the second flip. On the other hand, if we are selecting cards one at time without replacement from a standard deck of cards, the probability of selecting the queen of spades on the second draw clearly depends on what happens on the first draw. If the queen of spades was already picked, the probability of picking her on the second draw would be zero. Thus drawing the queen of spades on the second draw without replacement is not independent from drawing the queen of spades on the first draw.

That is, two events E and F are independent if

$$P(E|F) = P(E) \qquad \text{and} \qquad P(F|E) = P(F)$$

In words, the probability of E given that F has occurred is the same probability of E if F had not occurred. Similarly, the probability of F given that E has occurred is just the probability of F if E had not occurred.

Independent Events

Two events E and F are said to be independent if

$$P(E|F) = P(E) \qquad \text{and} \qquad P(F|E) = P(F)$$

We shall now obtain a result that is more convenient to apply when attempting to determine if two events are independent. It will also be useful when finding the probability that a series of independent events occurred.

If then two events E and F are independent, the previous comments together with the product rule indicate that

$$P(E \cap F) = P(E|F)P(F) = P(E)P(F)$$

Now consider the case that $P(E) > 0$ and $P(F) > 0$. Assume that $P(E \cap F) = P(E)P(F)$, then

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(E)P(F)}{P(F)} = P(E)$$

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{P(E)P(F)}{P(E)} = P(F)$$

This discussion then yields the following theorem.

Independent Events Theorem

Let E and F be two events with $P(E) > 0$ and $P(F) > 0$. Then E and F are independent if, and only if,

$$P(E \cap F) = P(E)P(F)$$

Although at times one is certain whether or not two events are independent, often one can only tell by doing the calculations.

EXAMPLE 6 **Smoking and Heart Disease** A study of 1000 men over 65 indicated that 250 smoked and 50 of these smokers had some signs of heart disease, while 100 of the nonsmokers showed some signs of heart disease. Let E be the event "smokes" and H be the event "has signs of heart disease." Are these two events independent?

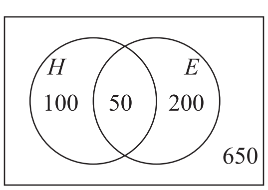


Figure 1.26

Solution The Venn diagram is given in Figure 1.26. From this diagram $P(H) = 0.15$, $P(E) = 0.25$, and $P(H \cap E) = 0.05$. Thus

$$P(H)P(E) = (0.15)(0.25) = 0.0375$$
$$P(H \cap E) = 0.05$$

Since $0.0375 \neq 0.05$, these two events are not independent. ✦

EXAMPLE 7 **Determining if Two Events Are Independent** In a medical trial a new drug was effective for 60% of the patients and 30% of the patients suffered from a side effect. If 28% of the patients did not find the drug effective nor had a side effect, are the events E (drug was effective) and F (patient had a side effect) independent?

Solution We can organize this information in a Venn diagram. If 28% of the patients did not find the drug effective nor had a side effect, this means that 0.28 is placed in the region outside the E and F circles. See Figure 1.27. The region inside will have the probability $(E \cup F) = 1 - 0.28 = 0.72$. Since we are not told how many found the drug effective and had a side effect, $P(E \cap F)$, we will need to use the union rule to find this number,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
$$0.72 = 0.6 + 0.3 - P(E \cap F)$$
$$P(E \cap F) = 0.9 - 0.72 = 0.18$$

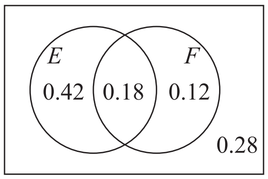


Figure 1.27

Place this in the Venn diagram in Figure 1.27. For completeness we could find $P(E \cap F^c) = 0.6 - 0.18 = 0.42$ and $P(E^c \cap F) = 0.3 - 0.18 = 0.12$ and place those in the diagram. However, to check for independence we need only place the three values needed in the formula:

$$P(E)P(F) = (0.60)(0.30) = 0.18$$
$$P(E \cap F) = 0.18$$

Since $0.18 = 0.18$, these events are independent. ✦

REMARK: Notice that saying two events are independent is not the same as saying that they are mutually exclusive. The sets in both previous examples were not mutually exclusive, but in one case the sets were independent and in the other case they were not.

The notion of independence can be extended to any number of finite events.

> **Independent Set of Events**
> A set of events $\{E_1, E_2, \ldots, E_n\}$ is said to be independent if, for any k of these events, the probability of the intersection of these k events is the product of the probabilities of each of the k events. This must hold for any $k = 2, 3, \ldots, n$.

For example, for the set of events $\{E, F, G\}$ to be independent all of the following must be true:

$$P(E \cap F) = P(E)P(F), \quad P(E \cap G) = P(E)P(G)$$

$$P(F \cap G) = P(F)P(G), \quad P(E \cap F \cap G) = P(E)P(F)P(G)$$

It is intuitively clear that if two events E and F are independent, then so also are E and F^c, E^c and F, E^c and F^c. (See Exercises 52 and 53.) Similar statements are true about a set of events.

EXAMPLE 8 **Independent Events and Safety** An aircraft has a system of three computers, each independently able to exercise control of the flight. The computers are considered 99.9% reliable during a routine flight. What is the probability of having a failure of the control system during a routine flight?

Solution Let the events E_i, $i = 1,2,3$ be the three events given by the reliable performance of respectively the first, second, and third computer. Since the set of events $\{E_1, E_2, E_3\}$ is independent, so is the set of events $\{E_1^c, E_2^c, E_3^c\}$. The system will fail only if all three computers fail. Thus the probability of failure of the system is given by

$$\{E_1^c \cap E_2^c \cap E_3^c\} = P(E_1^c)P(E_2^c)P(E_3^c) = (0.001)^3$$

which, of course, is an extremely small number. ✦

EXAMPLE 9 **Broken Elevators** A building has three elevators. The chance that elevator A is not working is 12%, the chance that elevator B is not working is 15% and the chance that elevator C is not working is 9%. If these probabilities are independent, what is the probability that exactly one elevator is not working?

Solution The probability that elevator A is not working but the other two are working is $(0.12)(0.85)(0.91)$. The probability that only elevator B is working is $(0.88)(0.15)(0.91)$ and only elevator C working has probability $(0.88)(0.85)(0.09)$ of occurring. The probability that exactly one does not work is the sum of these three probabilities:

$$\begin{aligned} P &= (0.12)(0.85)(0.91) + (0.88)(0.15)(0.91) + (0.88)(0.85)(0.09) \\ &= 0.28026 \end{aligned}$$ ✦

Self-Help Exercises 1.6

1. Three companies A, B, and C, are competing for a contract. The probabilities that they receive the contract are, respectively, $P(A) = 1/6$, $P(B) = 1/3$, and $P(C) = 1/2$. What is the probability that A will receive the contract if C pulls out of the bidding?

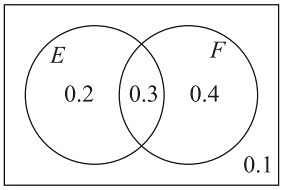
2. Two bins contain transistors. The first has 4 defective and 15 non-defective transistors, while the second has 3 defective and 22 non-defective ones. If the probability of picking either bin is the same, what is the probability of picking the second bin and a defective transistor?

3. Success is said to breed success. Suppose you are in a best of 3 game tennis match with an evenly matched opponent. However, if you win a game, your probability of winning the next increases from $1/2$ to $2/3$. Suppose, however, that if you lose, the probability of winning the next match remains the same. (Success does not breed success for your opponent.) What is your probability of winning the match? Hint: Draw a tree.

4. A family has three children. Let E be the event "at most one boy" and F the event "at least one boy and at least one girl." Are E and F independent if a boy is as likely as a girl? Hint: Write down every element in the sample space S and the events E, F, and $E \cap F$, and find the appropriate probabilities by counting.

1.6 Exercises

In Exercises 1 through 6, refer to the accompanying Venn diagram to find the conditional probabilities.



1. **a.** $P(E|F)$ **b.** $P(F|E)$

2. **a.** $P(E^c|F)$ **b.** $P(F^c|E)$

3. **a.** $P(E|F^c)$ **b.** $P(F|E^c)$

4. **a.** $P(E^c|F^c)$ **b.** $P(F^c|E^c)$

5. **a.** $P(F|E \cap F)$ **b.** $P(F^c|F)$

6. **a.** $P(E^c \cap F|F)$ **b.** $P(E \cap F^c|F)$

In Exercises 7 through 12, let $P(E) = 0.4$, $P(F) = 0.6$, and $P(E \cap F) = 0.2$. Draw a Venn diagram and find the conditional probabilities.

7. **a.** $P(E^c|F)$ **b.** $P(F^c|E)$

8. **a.** $P(E|F)$ **b.** $P(F|E)$

9. **a.** $P(F^c|E^c)$ **b.** $P(E \cup F|E)$

10. **a.** $P(E|F^c)$ **b.** $P(F|E^c)$

11. **a.** $P(E^c \cap |F)$ **b.** $P(E \cap F^c|E)$

12. **a.** $P(F|E \cap F)$ **b.** $P(E^c|E)$

In Exercises 13 through 20, determine if the given events E and F are independent.

13. $P(E) = 0.3$, $P(F) = 0.5$, $P(E \cap F) = 0.2$

14. $P(E) = 0.5$, $P(F) = 0.7$, $P(E \cap F) = 0.3$

15. $P(E) = 0.2$, $P(F) = 0.5$, $P(E \cap F) = 0.1$

16. $P(E) = 0.4$, $P(F) = 0.5$, $P(E \cap F) = 0.2$

17. $P(E) = 0.4$, $P(F) = 0.3$, $P(E \cup F) = 0.6$

18. $P(E \cap F^c) = 0.3$, $P(E \cap F) = 0.2$, $P(E^c \cap F) = 0.2$

19. $P(E \cap F^c) = 0.3$, $P(E \cap F) = 0.3$, $P(E^c \cap F) = 0.2$

20. $P(E) = 0.2$, $P(F) = 0.5$, $P(E \cup F) = 0.6$

21. A pair of fair dice is tossed. What is the probability that a sum of seven has been tossed if it is known that at least one of the numbers is a 3.

22. A single fair die is tossed. What is the probability that a 3 occurs on the top if it is known that the number is a prime?

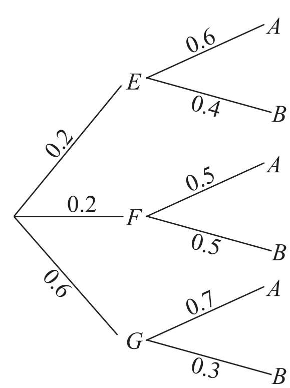
23. A fair coin is flipped three times. What is the probability that heads occurs three times if it is known that heads occurs at least once?

24. A fair coin is flipped four times. What is the probability that heads occurs three times if it is known that heads occurs at least twice?

25. Three cards are randomly drawn without replacement from a standard deck of 52 cards.
    - **a.** What is the probability of drawing an ace on the third draw?
    - **b.** What is the probability of drawing an ace on the third draw given that at least one ace was drawn on the first two draws?

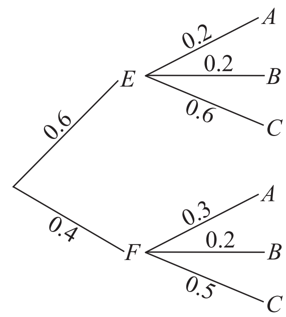
26. Three balls are randomly drawn from an urn that contains four white and six red balls.
    - **a.** What is the probability of drawing a red ball on the third draw?
    - **b.** What is the probability of drawing a red ball on the third draw given that at least one red ball was drawn on the first three draws?

27. From the tree diagram find
    a. $P(A \cap E)$ **b.** $P(A)$ **c.** $P(A|E)$



28. From the tree diagram find

a. $P(A \cap E)$ **b.** $P(A)$ **c.** $P(A|E)$



29. An urn contains five white, three red, and two blue balls. Two balls are randomly drawn. What is the probability that one is white and one is red if the balls are drawn

a. without replacement?

b. with replacement after each draw?

30. An urn contains four white and six red balls. Two balls are randomly drawn. If the first one is white, the ball is replaced. If the first one is red, the ball is not replaced. What is the probability of drawing at least one white ball?

31. In a family of four children, let E be the event "at most one boy" and F the event "at least one girl and at least one boy." If a boy is as likely as a girl, are these two events independent?

32. A fair coin is flipped three times. Let E be the event "at most one head" and F the event "at least one head and at least one tail." Are these two events independent?

33. The two events E and F are independent with $P(E) = 0.3$ and $P(F) = 0.5$. Find $P(E \cup F)$.

34. The two events E and F are independent with $P(E) = 0.4$ and $P(F) = 0.6$. Find $P(E \cup F)$.

35. The three events E, F, and G are independent with $P(E) = 0.2$, $P(F) = 0.3$, and $P(G) = 0.5$. What is $P(E \cup F \cup G)$?

36. The three events E, F, and G are independent with $P(E) = 0.3$, $P(F) = 0.4$, and $P(G) = 0.6$. What is $P(E^c \cup F^c \cup G^c)$?

Applications

37. Manufacturing A plant has three assembly lines with the first line producing 50% of the product and the second 30%. The first line produces defective products 1% of the time, the second line 2% of the time, and the third 3% of the time.

a. What is the probability that a defective product is produced at this plant given that it was made on the second assembly line?

b. What is the probability that a defective product is produced at this plant?

38. Manufacturing Two machines turn out all the products in a factory, with the first machine producing 40% of the product and the second 60%. The first machine produces defective products 2% of the time and the second machine 4% of the time.

a. What is the probability that a defective product is produced at this factory given that it was made on the first machine?

b. What is the probability that a defective product is produced at this factory?

39. Suppliers A manufacturer buys 40% of a certain part from one supplier and the rest from a second supplier. It notes that 2% of the parts from the first supplier are defective, and 3% are defective from the second supplier. What is the probability that a part is defective?

40. Advertising A television ad for a company's product has been seen by 20% of the population. Of those who see the ad, 10% then buy the product. Of those who do not see the ad, 2% buy the product. Find the probability that a person buys the product.

41. Reliability A firm is making a very expensive optical lens to be used in an earth satellite. To be assured that the lens has been ground correctly, three independent tests using entirely different techniques are used. The probability is 0.99 that any of one of these tests will detect a defect in the lens. What is the probability that the lens has a defect even though none of the three tests so indicates?

42. Psychology and Sales A door-to-door salesman expects to make a sale 10% of the time when starting the day. But making a sale increases his enthusiasm so much that the probability of a sale to the next customer is 0.2. If he makes no sale, the probability for a sale stays at 0.l. What is the probability that he will make at least two sales with his first three visits?

43. Quality Control A box contains two defective (D) parts and five non-defective (N) ones. You randomly select a part (without replacement) until you get a non-defective part. What is the probability that the number of parts selected is

a. one **b.** two **c.** three?

44. **Sales** A company sells machine tools to two firms in a certain city. In 40% of the years it makes a sale to the first firm, in 30% of the years to the second firm, and in 10% to both. Are the two events "a sale to the first firm" and "a sale to the second firm" independent?

45. **Medicine** In a study of 250 men over 65, 100 smoked, 60 of the smokers had some signs of heart disease, and 90 of the nonsmokers showed some signs of heart disease. Let E be the event "smokes" and H be the event "has signs of heart disease." Are these two events independent?

46. **Contracts** A firm has bids on two contracts. It is known that the awarding of these two contracts are independent events. If the probability of receiving the contracts are 0.3 and 0.4, respectively, what is the probability of not receiving either?

47. **Stocks** A firm checks the last 200 days on which its stock has traded. On 100 of these occasions the stock has risen in price with a broad-based market index also rising on 70 of these particular days. The same market index has risen 90 of the 200 trading days. Are the movement of the firm's stock and the movement of the market index independent?

48. **Bridges** Dystopia County has three bridges. In the next year, the Elder bridge has a 15% chance of collapse, the Younger bridge has a 5% chance of collapse and the Ancient bridge has a 20% chance of collapse. What is the probability that exactly one bridge will collapse in the next year?

49. **Missing Parts** A store sells desk kits. In each kit there is a 2% chance that a screw is missing, a 3% chance that a peg is missing, and a 1% chance that a page of directions is missing. If these events are independent, what is the probability that a desk kit has exactly one thing missing?

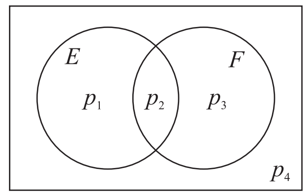
Extensions

50. **Medicine** The probability of residents of a certain town contracting cancer is 0.01. Let x be the percent of residents that work for a certain chemical plant and suppose that the probability of both working for this plant and of contracting cancer is 0.001. What must x be for the two events "gets cancer" and "work for the chemical plant" to be independent?

51. Given the probabilities shown in the accompanying Venn diagram, show that the events E and F are independent if, and only if,

$$p_1 p_3 = p_2 p_4$$

What must the Venn diagram look like if the sets are mutually disjoint?



52. Show that if E and F are independent, then so are E and F^c.

53. Show that if E and F are independent, then so are E^c and F^c.

54. Show that two events are independent if they are mutually exclusive and the probability of one of them is zero.

55. Show that two events E and F are not independent if they are mutually exclusive and both have nonzero probability.

56. Show that if E and F are independent events, then

$$P(E \cup F) = 1 - P(E^c)P(F^c)$$

57. If $P(F) > 0$, then show that

$$P(E^c|F) = 1 - P(E|F)$$

58. If E, F, and G are three events and $P(G) > 0$, show that

$$P(E \cup G|G) = P(E|G) + P(F|G) - P(E \cap F|G)$$

59. If E and F are two events with $F \subset E$, then show that $P(E|F) = 1$.

60. If E and F are two events with $E \cap F = \emptyset$, then show that $P(E|F) = 0$.

61. If E and F are two events, show that

$$P(E|F) + P(E^c|F) = 1$$

Solutions to Self-Help Exercises 1.6

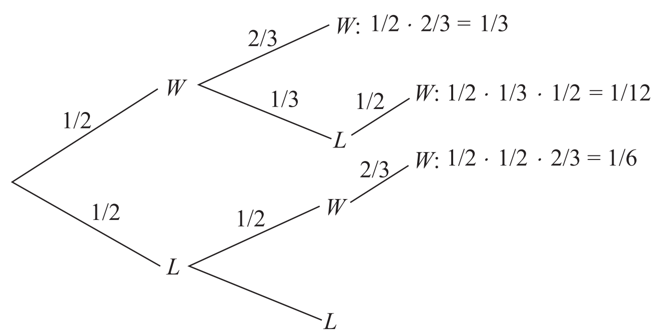
1. If E is the event that A obtains the contract and F the event that either A or B obtain the contract, then $E = \{A\}$, $F = \{A, B\}$, and $E \cap F = \{A\}$, and the conditional probability that A will receive the contract if C pulls out of the bidding is

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/6}{1/2} = \frac{1}{3}$$

2. The sample space is $S = \{1D, 1N, 2D, 2N\}$ where the number refers to picking the first or second bin and the letter refers to picking a defective (D) or non-defective (N) transistor. If E is picking the first bin and F is picking a defective transistor then

$$P(E \cap F) = P(E)P(F|E) = \frac{1}{2} \cdot \frac{3}{25} = \frac{3}{50}$$

3. The appropriate tree is given.



The probability of winning the match is then

$$\frac{1}{3} + \frac{1}{12} + \frac{1}{6} = \frac{7}{12}$$

4. The elements in the spaces S, E, and F are

$$S = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}$$
$$E = \{BGG, GBG, GGB, GGG\}$$
$$F = \{BBG, BGB, BGG, GBB, GBG, GGB\}$$

With $E \cap F = \{BGG, GBG, GGB\}$. Thus counting elements gives

$$P(E \cap F) = \frac{3}{8} \qquad P(E)P(F) = \frac{4}{8} \cdot \frac{6}{8} = \frac{3}{8}$$

Since these two numbers are the same, the two events are independent.

1.7 Bayes' Theorem

APPLICATION
Probability of a Defective Part

Recall Example 4 of the last section. A company makes the components for a product at a central location. These components are shipped to three plants, 1, 2, and 3, for assembly into a final product. The percentages of the product assembled by the three plants are, respectively, 50%, 20%, and 30%. The percentages of defective products coming from these three plants are, respectively, 1%, 2%, and 3%. Given a defective product, what is the probability it was assembled at Plant 1? At Plant 2? At Plant 3? See Example 1 for the answers to these questions.

✧ Bayesian Reasoning

We have been concerned with finding the probability of an event that will occur in the future. We now look at calculating probabilities after the events have occurred. This is a surprisingly common and important application of conditional probability. For example, when an e-mail message is received, the e-mail program does not know if the message is junk or not. But, given the number of misspelled words or the presence of certain keywords, the message is probably junk and is appropriately filtered. This is called Bayesian filtering and knowledge of what happened second (the misspelled words) allows an estimate of what happened first (the mail is junk).

We will now consider a situation where we will use Bayesian reasoning to estimate what happened in the first part of an experiment when you only know the results of the second part. Your friend has a cup with five green marbles and two red marbles and a bowl with one green, and three red, as shown in Figure 1.28a. There is an equal chance of choosing the cup or the bowl. After the cup or bowl has been chosen, a marble is selected from the container. A tree diagram for this experiment is shown in Figure 1.28b. Notice that on the branches of the tree diagram we have the conditional probabilities such as $P(G|C)$ which is the probability you choose a green marble given you are choosing from the cup.

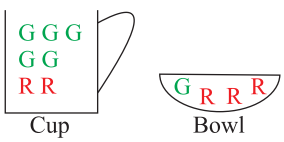


Figure 1.28a

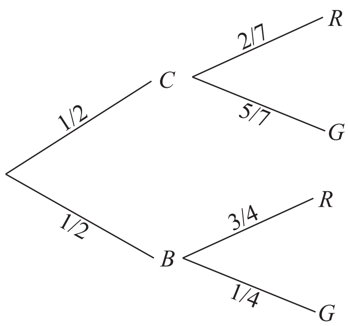


Figure 1.28b

Now your friend hides the bowl and cup and performs the experiment. He then shows you he has picked a green marble and asks, "what is the probability this green marble came from the cup"? That is, what is $P(C|G)$? Notice this is **not** what we have on the tree diagram. How can we figure this out?

Intuitively, if you had to guess if it was more likely to have come from the cup or the bowl, you would say it came from the cup. Looking at the figure, the cup has relatively more green ones, and if it was equally likely to have been picked from the cup or bowl, given it is green, more likely than not it came from the cup. But how can we get an exact value for the probability? Recall the formula used in the previous section for conditional probability,

$$P(C|G) = \frac{P(C \cap G)}{P(G)}$$

We can find $P(C \cap G)$ from the product rule. From multiplying along the branches it is

$$P(C \cap G) = P(C) \cdot P(G|C) = \frac{1}{2} \cdot \frac{5}{7} = \frac{5}{14}$$

What about the denominator, $P(G)$? We can get green marbles from the cup or green marbles from the bowl. The total probability will be the sum of the probabilities of those two mutually exclusive events:

$$P(G) = P(G \cap C) + P(G \cap B) = \frac{1}{2}\cdot\frac{5}{7} + \frac{1}{2}\cdot\frac{1}{4} = \frac{27}{56}$$

.

Putting the pieces together we have

$$P(C|G) = \frac{5/14}{27/56} = \frac{20}{27} \approx 0.74$$

So there was about a 74% chance that the marble came from the cup, given that it was green.

This is an example of **Bayes' theorem**. This theorem was discovered by the Presbyterian minister Thomas Bayes (1702–1763). We now state this in a more general form.

✧ Bayes' Theorem

We suppose that we are given a sample space S and three mutually exclusive events E_1, E_2, and E_3, with $E_1 \cup E_2 \cup E_3 = S$ as indicated in Figure 1.29. Notice that the three events divide the same space S into 3 partitions. Given another event F, the tree diagram of possibilities is shown in Figure 1.30.

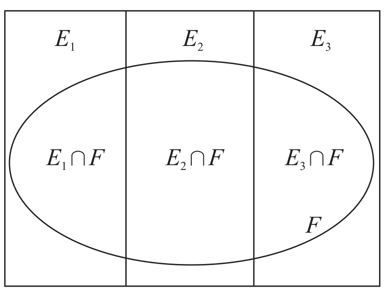


Figure 1.29

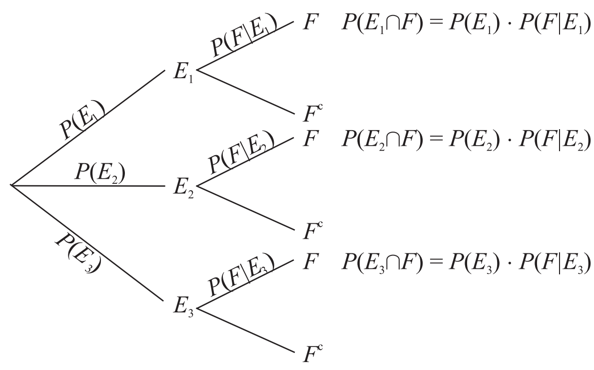


Figure 1.30

The probability $P(F|E_1)$ is the fraction with the numerator given by the probability off the branch $E_1 \cap F$ while the denominator is the sum of all the probabilities of all branches that end in F. This is

$$P(E_1|F) = \frac{P(E_1)P(F|E_1)}{P(E_1)P(F|E_1) + P(E_2)P(F|E_2) + P(E_3)P(F|E_3)}$$

The same for $P(F|E_2)$.

$$P(E_2|F) = \frac{P(E_2)P(F|E_2)}{P(E_1)P(F|E_1) + P(E_2)P(F|E_2) + P(E_3)P(F|E_3)}$$

and so on. We then state Bayes' theorem in even more general form.

Bayes' Theorem

Let $E_1, E_2, \ldots, E_n$, be mutually exclusive events in a sample space S with $E_1 \cup E_2 \cup \ldots \cup E_n = S$. If F is any event in S, then for $i = 1, 2, \ldots, n$,

$$P(E_i|F) = \frac{\text{probability of branch } E_i \cap F}{\text{sum of all probabilities of all branches that end in } F}$$
$$= \frac{P(E_i)P(F|E_i)}{P(E_1)P(F|E_1) + P(E_2)P(F|E_2) + \cdots + P(E_n)P(F|E_n)}$$

REMARK: The formula above looks quite complex. However, Bayes' theorem is much easier to remember by simply keeping in mind the original formula is simply $P(E|F) = {}^{P(E\cap F)}/_{P(F)}$ and that sometimes to find $P(F)$ you need to add together all the different ways that F can occur.

EXAMPLE 1 **Defective Components** At the start of this section a question was posed to find the probability that a defective component came from a certain plant. Find these probabilities.

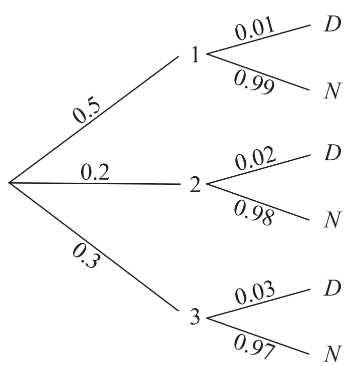


Figure 1.31

Solution Refer to Figure 1.31 for the probabilities that are given. Using the product rule we have

$$P(1 \cap D) = P(1)P(D|1) = (0.5)(0.01) = 0.005$$
$$P(2 \cap D) = P(2)P(D|2) = (0.2)(0.02) = 0.004$$
$$P(3 \cap D) = P(3)P(D|3) = (0.3)(0.03) = 0.009$$

and thus

$$P(D) == P(1 \cap D) + P(2 \cap D) + P(3 \cap D)$$
$$= 0.005 + 0.004 + 0.009$$
$$= 0.018$$

Using this information and the definition of conditional probability, we then have

$$P(1|D) = \frac{P(1 \cap D)}{P(D)} = \frac{0.005}{0.005 + 0.004 + 0.009} = \frac{0.005}{0.018} = \frac{5}{18}$$

We now notice that the numerator of this fraction is the probability of the branch $1 \cap D$, while the denominator is the sum of all the probabilities of all branches that end in D. We also have

$$P(2|D) = \frac{P(2 \cap D)}{P(D)} = \frac{0.004}{0.018} = \frac{4}{18}$$

Now notice that the numerator of this fraction is the probability of the branch $2 \cap D$, while the denominator is the sum of all the probabilities of all branches that end in D. A similar statement can be made for

$$P(3|D) = \frac{P(3 \cap D)}{P(D)} = \frac{0.009}{0.018} = \frac{9}{18}$$ ✦

A very interesting example occurs in medical tests for disease. All medical tests have what are called false positives and false negatives. That is, a test result could come back positive and the patient does not have the disease or a test could come back negative and the patient does have the disease. Many modern tests have low rates of false positives and false negatives, but even then there can be difficulties with a diagnosis. We begin with a test that gives every appearance of being excellent, but an important consequence may be disappointing.

EXAMPLE 2 **A Medical Application of Bayes' Theorem** The standard tine test for tuberculosis attempts to identify carriers, that is, people who have been infected by the tuberculin bacteria. The probability of a false negative is 0.08, that is, the probability of the tine test giving a negative reading to a carrier is $P(-|C) = 0.08$. The probability of a false positive is 0.04, that is, the probability of the tine test giving a positive indication when a person is a non-carrier is $P(+|N) = 0.04$.

The probability of a random person in the United States having tuberculosis is 0.0075. Find the probability that a person is a carrier given that the tine test gives a positive indication.

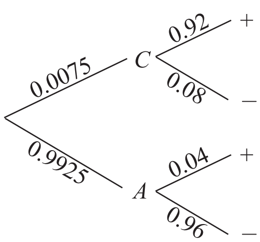


Figure 1.32

Solution The probability we are seeking is $P(C|+)$. Figure 1.32 shows the appropriate diagram where C is the event "is a carrier," N the event "is a non-carrier," $+$ the event "test yields positive result," and $-$ is the event "test is negative."

Then Bayes' theorem can be used and $P(C|+)$ is the probability of branch $C \cap +$ divided by the sum of all probabilities that end in $+$. Then

$$P(C|+) = \frac{P(C \cap +)}{P(+)} = \frac{P(C)P(+|C)}{P(C)P(+|C) + P(N)P(+|N)}$$
$$= \frac{(0.0075)(0.92)}{(0.0075)(0.92) + (0.9925)(0.04)} \approx 0.15$$

So only 15% of people with positive tine test results actually carry TB. ✦

REMARK: This number is surprisingly low. Does this indicate that the test is of little value? As Self-Help Exercise 1 will show, a person whose tine test is negative has a probability of 0.999 of not having tuberculosis. Such an individual can feel safe. The individuals whose tine test is positive are probably all right also but will need to undergo further tests, such as a chest x-ray.

In some areas of the United States the probability of being a carrier can be as high as 0.10. The following example examines the tine test under these conditions.

EXAMPLE 3 **A Medical Application of Bayes' Theorem** Using the information found in Example 2, find $P(C|+)$ again when $P(C) = 0.10$.

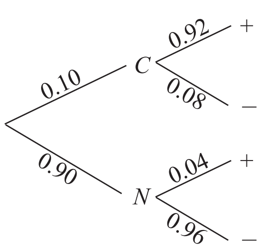


Figure 1.33

Solution See Figure 1.33 for the tree diagram with the probabilities. Using Bayes' theorem exactly as before, we obtain

$$P(C|+) = \frac{P(C)P(+|C)}{P(C)P(+|C) + P(N)P(+|N)}$$
$$= \frac{(0.10)(0.92)}{(0.10)(0.92) + (0.90)(0.04)} \approx 0.72$$

Thus 72% of these individuals who have a positive tine test result are carriers. ✦

Thus in the first example, when the probability of being a carrier is low, the tine test is useful for determining those who do not have TB. In the second example, when the probability of being a carrier is much higher, the tine test is useful for determining those who are carriers, although, naturally, these latter individuals will undergo further testing.

EXAMPLE 4 An Application of Bayes' Theorem Suppose there are only four economic theories that can be used to predict expansions and contractions in the economy. By polling economists on their beliefs on which theory is correct, the probability that each of the theories is correct has been determined as follows:

$$P(E_1) = 0.40, P(E_2) = 0.25, P(E_3) = 0.30, P(E_4) = 0.05$$

The economists who support each theory then use the theory to predict the likelihood of a recession (R) in the next year. These are as follows:

$$P(R|E_1) = 0.01, P(R|E_2) = 0.02, P(R|E_3) = 0.03, P(R|E_4) = 0.90$$

Now suppose a recession actually occurs in the next year. How would the probabilities of the correctness of the fourth and first theories be changed?

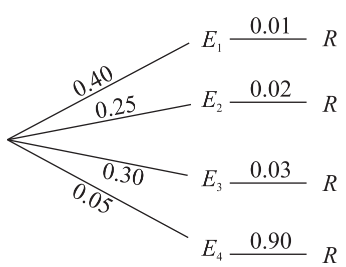


Figure 1.34

Solution We first note that the fourth theory E_4, has initially a low probability of being correct. Also notice that this theory is in sharp disagreement with the other three on whether there will be a recession in the next year.

Bayes' theorem gives $P(E_4|R)$ the probability of the branch $E_4 \cap R$ divided by the sum of all the probabilities of all the branches that end in R. See Figure 1.34. Similarly for the other theories. Thus $P(E_4|R)$ and and $P(E_1|R)$ are

$$\begin{aligned}
P(E_4|R) &= \frac{P(E_4 \cap R)}{P(R)} \\
&= \frac{P(E_4)P(R|E_4)}{P(E_1)P(R|E_1) + P(E_2)P(R|E_2) + P(E_3)P(R|E_3) + P(E_4)P(R|E_4)} \\
&= \frac{(0.05)(0.90)}{(0.40)(0.01) + (0.25)(0.02) + (0.30)(0.03) + (0.05)(0.90)} \\
&= \frac{0.045}{0.063} \approx 0.71
\end{aligned}$$

$$\begin{aligned}
P(E_1|R) &= \frac{P(E_1 \cap R)}{P(R)} \\
&= \frac{P(E_1)P(R|E_1)}{P(E_1)P(R|E_1) + P(E_2)P(R|E_2) + P(E_3)P(R|E_3) + P(E_4)P(R|E_4)} \\
&= \frac{(0.04)(0.10)}{(0.40)(0.01) + (0.25)(0.02) + (0.30)(0.03) + (0.05)(0.90)} \\
&= \frac{0.004}{0.063} \approx 0.06
\end{aligned}$$

Thus, given that the recession did occur in the next year, the probability that E_4 is correct has jumped, while the probability that E_1 is true has plunged. ✦

Although this is an artificial example probabilities indeed are revaluated in this way based on new information.

Self-Help Exercises 1.7

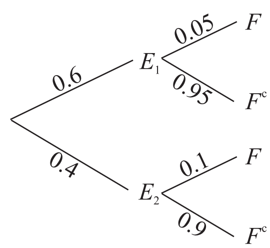
1. Referring to Example 2 of the text, find the probability that an individual in the United States is not a carrier given that the tine test is negative.

2. A gym has three trainers, Aldo, Bertha, and Coco, who each have $1/3$ of the new members. A person who trains with Aldo has a 60% chance of being successful, a person who trains with Bertha has a 30% chance of being successful and a person who trains with Coco has a 90% chance of being successful. What is the probability that a successful member had Bertha as her trainer?

3. A purse has three nickels and five dimes. A wallet has two nickels and one dime. A coin is chosen at random from the purse and placed in the wallet. A coin is then drawn from the wallet. If a dime is chosen from the wallet, what is the probability that the transferred coin was a nickel?

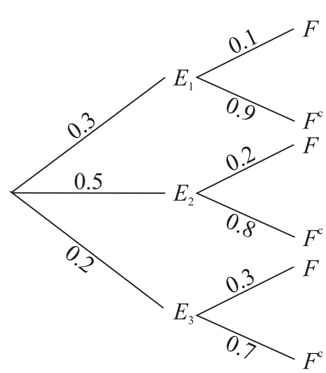
1.7 Exercises

1. Find $P(E_1|F)$ and $P(E_2|F^c)$ using the tree diagram below.



2. Find $P(E_1|F^c)$ and $P(E_2|F)$ using the tree diagram in the previous exercise.

3. Find $P(E_1|F)$ and $P(E_1|F^c)$ using the tree diagram below.



4. Find $P(E_3|F)$ and $P(E_3|F^c)$ using the tree diagram in the previous exercise.

Exercises 5 through 8 refer to two urns that each contain 10 balls. The first urn contains 2 white and 8 red balls. The second urn contains 7 white and 3 red balls. An urn is selected, and a ball is randomly drawn from the selected urn. The probability of selecting the first urn is $2/3$.

5. If the ball is white, find the probability that the first urn was selected.

6. If the ball is white, find the probability that the second urn was selected.

7. If two balls are drawn from the selected urn without replacement and both are white, what is the probability that the urn selected was the
   a. the first one?
   b. the second one?

8. A ball is drawn from the selected urn and replaced. Then another ball is drawn and replaced from the same urn. If both balls are white, what is the probability that the urn selected was
   a. the first one?
   b. the second one?

Exercises 9 through 12 refer to three urns that each contain 10 balls. The first contains 2 white and 8 red balls, the second 5 white and 5 red, and the third all 10 white. Each urn has an equal probability of being selected. After an urn is selected a ball is randomly drawn from this urn.

9. If the ball drawn was white, find the probability that the first urn was selected.

10. If the ball drawn was white, find the probability that the third urn was selected.

11. Suppose two balls are drawn from the selected urn without replacement and both are white. What is the probability that the urn selected was

a. the first one? **b.** the second one?

12. Now a ball is drawn, replaced, and then another drawn. Suppose both are white. What is the probability that the urn selected was

a. the first one? **b.** the second one?

Exercises 13 through 16 refer to the following experiment: A box has two blue and six green jelly beans. A bag has five blue and four green jelly beans. A jelly bean is selected at random from the box and placed in the bag. Then a jelly bean is selected at random from the bag.

13. If a blue jelly bean is selected from the bag, what is the probability that the transferred jelly bean was blue?

14. If a green jelly bean is selected from the bag, what is the probability that the transferred jelly bean was green?

15. If a green jelly bean is selected from the bag, what is the probability that the transferred jelly bean was blue?

16. If a blue jelly bean is selected from the bag, what is the probability that the transferred jelly bean was green?

Exercises 17 through 20 refer to the following experiment: Two cards are drawn in succession without replacement from a standard deck of 52 playing cards.

17. What is the probability that the first card drawn was a spade given that the second card drawn was not a spade?

18. What is the probability that the first card drawn was a queen given that the second card drawn was not a queen?

19. What is the probability that the first card drawn was a heart given that the second card drawn was a diamond?

20. What is the probability that the first card drawn was an ace given that the second card drawn was a king?

Applications

21. **Manufacturing** A plant has three assembly lines with the first line producing 50% of the product and the second 30%. The first line produces defective products 1% of the time, the second line 2% of the time, and the third 3% of the time. Given a defective product, what is the probability it was produced on the second assembly line? See Exercise 37 of the previous section.

22. **Manufacturing** Two machines turn out all the products in a factory, with the first machine producing 40% of the product and the second 60%. The first machine produces defective products 2% of the time and the second machine 4% of the time. Given a defective product, what is the probability it was produced on the first machine? See Exercise 38 of the previous section.

23. **Medicine** Do Example 2 of the text if all the information remains the same except that the tine test has the remarkable property that $P(+|C)=1$. Compare your answer to the one in Example 2.

24. **Medicine** Using the information in Example 2 of the text, find $P(N|-)$, where $-$ is the event "test shows negative."

25. **Economics** Using the information in Example 4 of the text, find $P(E_1|R^c)$ and $P(E_4|R_c)$. Compare your answers with $P(E_1)$ and $P(E_4)$.

26. **Manufacturing** For Example 1, find $P(E_1|F^c)$.

27. **Quality Control** One of two bins is selected at random, one as likely to be selected as the other, and from the bin selected a transistor is chosen at random. The transistor is tested and found to be defective. It is known that the first bin contains two defective and four nondefective transistors, while the second bin contains five defective and one nondefective transistors. Find the probability that second bin was selected.

28. **Quality Control** Suppose in the previous exercise there is a third bin with five transistors, all of which are defective. Now one of the three bins is selected at random, one as likely to be selected as any other, and from this bin a transistor is chosen at random. If the transistor is defective find the probability it came from the third one.

29. **Quality Control** A typical box of 100 transistors contains only 1 defective one. It is realized that among the last 10 boxes, one box has 10 defective transistors. An inspector picks a box at random, and the first transistor selected is found to be defective. What is the probability that this box is the bad one?

30. Quality Control A typical box of 100 transistors contains only 1 defective one. It is realized that among the last 10 boxes, one box has 10 defective transistors. An inspector picks a box at random, inspects two transistors from this box on a machine and discovers that one of them is defective and one is not. What is the probability that this box is the bad one?

31. Manufacturing A manufacturing firm has four machines that produce the same component. Using the table, given that a component is defective find the probability that the defective component was produced by

a. machine 1 **b.** machine 2.

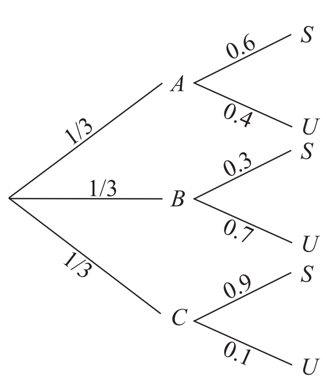
Machine	Percentage of Components Produced	Percentage of Defective Components
1	20	1
2	30	2
3	40	3
4	10	4

32. Social Sciences A man claimed not to be the father of a certain child. On the basis of evidence presented, the court felt that this man was twice as likely to be the father as not, and, hardly satisfied with these odds, required the man to take a blood test. The mother of the child had a different blood type than the child: therefore the blood type of the child was completely determined by the father. If the man's blood type was different than the child, then he could not be the father. The blood type of the child occurred in only 10% of the population. The blood tests indicated that the man had the same blood type as the child. What is the probability that the man is the father?

33. Medical Diagnosis A physician examines a patient and, on the basis of the symptoms, determines that he may have one of four diseases; the probability of each is given in the table. She orders a blood test, which indicates that the blood is perfectly normal. Data are available on the percentage of patients with each disease whose blood tests are normal. On the basis of the normal blood test, find all probabilities that the patient has each disease.

Diseases	Probability of Disease of Disease	Percentage of Normal Blood With This Disease
1	0.1	60
2	0.2	20
3	0.3	20
4	0.4	10

Solutions to Self-Help Exercises 1.7



1. Bayes' theorem can be used, and $F = -$. Then

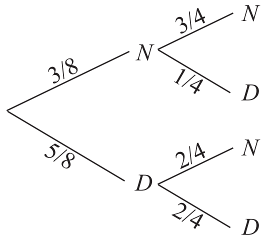
$$P(N|-) = \frac{P(N \cap -)}{P(-)} = \frac{P(N)P(-|N)}{P(N)P(-|N) + P(C)P(-|C)}$$

$$= \frac{(0.9925)(0.96)}{(0.9925)(0.96) + (0.0075)(0.08)} \approx 0.999$$

2. Begin with a tree diagram as shown in the figure on the left where S is a successful member and U is an unsuccessful member. To find $P(B|S)$ use Bayes' theorem.

$$P(B|S) = \frac{P(B \cap S)}{P(S)} = \frac{\frac{1}{3} \cdot 0.3}{\frac{1}{3} \cdot 0.6 + \frac{1}{3} \cdot 0.3 + \frac{1}{3} \cdot 0.9}$$

$$= \frac{0.1}{0.2 + 0.1 + 0.3} = \frac{1}{6} \approx 0.1667$$



3. Begin with a tree diagram. On the first set of branches we show the selection of a coin from the purse. Following the top branch, the nickel is placed in the wallet. Now the wallet has three nickels and one dime and a coin is chosen. Following the lower branch we place a dime in the wallet. The wallet then has two nickels and two dimes from which a coin is chosen.

We are asked to find $P(N_1|D_2)$ so we use Bayes' theorem:

$$P(N_1|D_2) = \frac{P(N_1 \cap D_2)}{P(D_2)} = \frac{P(N_1 \cap D_2)}{P(N_1 \cap D_2) + P(D_1 \cap D_2)}$$

$$= \frac{\frac{3}{8} \cdot \frac{1}{4}}{\frac{3}{8} \cdot \frac{1}{4} + \frac{5}{8} \cdot \frac{2}{4}} = \frac{3}{13}$$

Review

✧ Summary Outline

- If every element of a set A is also an element of another set B, we say that A is a **subset** of B and write $A \subseteq B$. If A is not a subset of B, we write $A \nsubseteq B$.
- If every element of a set A is also an element of another set B and $A \neq B$, we say that A is a **proper subset** of B and write $A \subset B$. If A is not a proper subset of B, we write $A \not\subset B$.
- The **empty set**, written as $\emptyset$, is the set with no elements.
- Given a universal set U and a set $A \subset U$, the **complement** of A, written A^c, is the set of all elements that are in U but not in A, that is,

$$A^c = \{x | x \in U, x \notin A\}$$

- The **union** of two sets A and B, written $A \cup B$, is the set of all elements that belong to A, or to B, or to both. Thus

$$A \cup B = \{x | x \in A \text{ or } x \in B \text{ or both}\}$$

- The **intersection** of two sets A and B, written $A \cap B$, is the set of all elements that belong to both the set A and to the set B.
- If $A \cap B = \emptyset$ then the sets A and B are **disjoint**.
- **Rules for Set Operations**

$A \cup B = B \cup A$	Commutative law for union
$A \cap B = B \cap A$	Commutative law for intersection
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative law for union
$A \cap (B \cap C) = (A \cap B) \cap C)$	Associative law for intersection
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive law for union
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive law for intersection
$(A \cup B)^c = A^c \cap B^c$	De Morgan law
$(A \cap B)^c = A^c \cup B^c$	De Morgan law

- If A is a set with a finite number of elements, we denote the number of elements in A by $n(A)$
- If the sets A and B are disjoint, then $n(A \cup B) = n(A) + n(B)$.
- For any finite sets A and B we have the **union rule**,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

- An **experiment** is an activity that has observable results. The results of the experiment are called **outcomes**.
- A **sample space** of an experiment is the set of all possible outcomes of the experiment. Each repetition of an experiment is called a **trial**.
- Given a sample space S for an experiment, an **event** is any subset E of S. An **elementary** event is an event with a single outcome.
- If E and F are two events, then $E \cup E$ is the **union** of the two events and consists of the set of outcomes that are in E or F or both.
- If E and F are two events, then $E \cap F$ is the **intersection** of the two events and consists of the set of outcomes that are in both E and F.
- If E is an event, then E^c is the **complement** of E and consists of the set of outcomes that are not in E.
- The empty set, $\emptyset$, is called the **impossible event**.
- Let S be a sample space. The event S is called the **certainty event**.
- Two events E and F are said to be **mutually exclusive** or **disjoint** if $E \cap F = \emptyset$.
- **Properties of Probability** Let S be a sample space and E, A, and B be events in S, $P(E)$ the probability of E, and so on. Then

  $0 \leq P(E) \leq 1$

  $P(S) = 1$

  $P(\emptyset) = 0$

  $P(A \cup B) = P(A) + P(B)$, if $A \cap B = \emptyset$

  $P(E^c) = 1 - P(E)$

  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  $P(A) \leq P(B)$, if $A \subset B$

- Let $P(E)$ be the probability of E, then the **odds for** E are

$$\frac{P(E)}{1 - P(E)} \text{ if } P(E) \neq 1$$

  This fraction reduced to lowest terms is $\frac{a}{b}$ and the odds are a:b.

- If the odds for an event E occurring is given as $\frac{a}{b}$ or a:b, then

$$P(E) = \frac{a}{a+b}$$

- If S is a finite uniform sample space and E is any event, then

$$P(E) = \frac{\text{Number of elements in } E}{\text{Number of elements in } S} = \frac{n(E)}{n(S)}$$

- Let E and F be two events in a sample space S. The **conditional probability** that E occurs given that F has occurred is defined to be

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \quad \text{if} \quad P(F) > 0$$

- Product rule: If E and F are two events in a sample space S with $P(E) > 0$ and $P(F) > 0$, then

$$P(E \cap F) = P(F)P(E|F) = P(E)P(F|E)$$

- Two events E and F are said to be **independent** if

$$P(E|F) = P(E) \quad \text{and} \quad P(F|E) = P(F)$$

- Let E and F be two events with $P(E) > 0$ and $P(F) > 0$. Then E and F are independent if, and only if, $P(E \cap F) = P(E)P(F)$.
- A set of events $\{E_1, E_2, \ldots, E_n\}$ is said to be independent if, for any k of these events, the probability of the intersection of these k events is the product of the probabilities of each of the k events. This must hold for any $k = 2, 3, \ldots, n$.
- **Bayes' Theorem** Let $E_1, E_2, \ldots, E_n$, be mutually exclusive events in a sample space S with $E_1 \cup E_2 \cup \ldots \cup E_n = S$. If F is any event in S, then for $i = 1, 2, \ldots, n$,

$$\begin{aligned} P(E_i|F) &= \frac{\text{probability of branch } E_iF}{\text{sum of all probabilities of all branches that end in } F} \\ &= \frac{P(E_i \cap F)}{P(F)} \\ &= \frac{P(E_i)P(F|E_i)}{P(E_1)P(F|E_1) + P(E_2)P(F|E_2) + \cdots + P(E_n)P(F|E_n)} \end{aligned}$$

✧ Review Exercises

1. Determine which of the following are sets:

a. current members of the board of Bank of America

b. past and present board members of Bank of America that have done an outstanding job

c. current members of the board of Bank of America who are over 10 feet tall

2. Write in set-builder notation:

$\{5, 10, 15, 20, 25, 30, 35, 40\}$.

3. Write in roster notation: $\{x | x^3 - 2x = 0\}$

4. List all the subsets of $\{A, B, C\}$.

5. On a Venn diagram indicate where the following sets are:

a. $A \cap B \cap C$ **b.** $A^c \cap B \cap C$ **c.** $(A \cup B)^c \cap C$

6. Let $U = \{1,2,3,4,5,6\}$, $A = \{1,2,3\}$, $B = \{2,3,4\}$, and $C = \{4,5\}$. Find the following sets: $A \cup B$, $A \cap B$, B^c, $A \cap B \cap C$, $(A \cup B) \cap C$, $A \cap B^c \cap C$

7. Let U be the set of all your current instructors and let

$H = \{x | x \text{ is at least 6 feet tall}\}$
$M = \{x | x \text{ is a male}\}$
$W = \{x | x \text{ weighs more than 180 pounds}\}$

Describe each of the following sets in words:

a. H^c **b.** $H \cup M$
c. $M^c \cap W^c$ **d.** $H \cap M \cap W$
e. $H^c \cap M \cap W$ **f.** $(H \cap M^c) \cup W$

8. Using the set H, M, and W in the previous exercise and set operations, write the set that represents the following statements:

a. my current female instructors
b. my current female instructors who weigh at most 180 pounds
c. my current female instructors who are at least 6 feet tall or else weigh more than 180 pounds

9. For the sets given in Exercise 6, verify that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

10. Use a Venn diagram to show that $(A \cap B)^c = A^c \cup B^c$

11. If $n(A) = 100$, $n(B) = 40$, and $n(A \cap B) = 20$, find $n(A \cup B)$.

12. If $n(A) = 40$ and $n(A \cap B^c) = 30$, find $n(A \cap B)$.

13. In a consumer advertising survey of 100 men, it was found that 20 watched the first game of the World Series, 15 watched the first game of the World Series and also watched the Super Bowl, while 30 did not watch either. How many watched the Super Bowl but not the first game of the World Series.

14. A consumer survey of 100 children found that

- 57 had a Barbie doll
- 68 had a teddy bear
- 11 had a toy piano
- 45 had a Barbie doll and a teddy bear
- 8 had a teddy bear and a toy piano
- 7 had a Barbie doll and a toy piano
- 5 had all three

a. How many had a Barbie doll and a teddy bear but not a toy piano?
b. How many had exactly 2 of these toys?
c. How many had none of these toys?

15. During a recent four-round golf tournament the number of strokes were recorded on a par 5 hole. The following table lists the frequencies of each number of strokes.

Strokes	3	4	5	6	7	8
Frequency	4	62	157	22	4	1

a. Find the probability for each number of strokes.
b. Find the probability that the number of strokes were less or equal to 5.
c. Find the probability that the number of strokes were less than 5.

16. An urn has 10 white, 5 red, and 15 blue balls. A ball is drawn at random. What is the probability that the ball will be

a. red? **b.** red or white? **c.** not white?

17. If E and F are disjoint sets in a sample space S with $P(E) = 0.25$ and $P(F) = 0.35$, find

a. $P(E \cup F)$ **b.** $n(E \cap F)$ **c.** $P(E^c)$

18. If E and F are two events in the sample space S with $P(E) = 0.20$, $P(F) = 0.40$, and $P(E \cap F) = 0.05$, find

a. $P(E \cup F)$ **b.** $P(E^c \cap F)$ **c.** $P((E \cup F)^c)$

19. Consider the sample space $S = \{a,b,c,d\}$ and suppose that $P(a) = P(b)$, $P(c) = P(d)$, and $P(d) = 2P(a)$. Find $P(b)$.

20. If the odds for a company obtaining a certain contract are 3:1, what is the probability that the company will receive the contract?

21. A furniture manufacturer notes the 6% of its reclining chairs have a defect in the upholstery, 4% a defect in the reclining mechanism, and 1% have both defects.

a. Find the probability that a recliner has at least one of these defects.
b. Find the probability that a recliner has none of these defects.

22. A survey of homeowners indicated that during the last year: 22% had planted vegetables, 30% flowers, 10% trees, 9% vegetables and flowers, 7% vegetables and trees, 5% flowers and trees, and 4%, all three of these.

a. Find the probability that a homeowner planted vegetables but not flowers.

b. Find the probability that exactly two of the items were planted.

c. Find the probability that none of these three items were planted.

23. Let $P(E) = 0.3$, $P(F) = 0.5$, and $P(E \cap F) = 0.2$. Draw a Venn diagram and find the indicated conditional probabilities:

a. $P(E|F)$ **b.** $P(E^c|F)$ **c.** $P(F^c|E^c)$

24. If $P(E) = 0.5$, $P(F) = 0.6$, and $P(E \cap F) = 0.4$, determine if E and F are independent events.

25. Reliability A spacecraft has three batteries that can operate all systems independently. If the probability that any battery will fail is 0.05, what is the probability that all three will fail?

26. Basketball A basketball player sinks a free throw 80% of the time. If she sinks one, the probability of sinking the next goes to 0.90. If she misses, the probability of sinking the next goes to 0.70. Find the probability that she will sink exactly two out of three free throws.

27. Manufacturing A manufacturing firm has 5 machines that produce the same component. Using the table, find the probability that a defective component was produced by

a. machine 1 **b.** machine 4

Machine	Components Produced	Defective Components
1	20%	1%
2	30%	2%
3	30%	3%
4	10%	4%
5	10%	10%

28. Drug Testing A company tests its employees for drug usage with a test that gives a positive reading 95% of the time when administered to a drug user and gives a negative reading 95% of the time when administered to a non-drug user. If 5% of the employees are drug users, find the probability that an employee is a non-drug user given that this person had a positive reading on the test. (The answer is shocking and illustrates the care that must be exercised in using such tests in determining guilt.)

✧ Project: Venn Diagrams and Systems of Linear Equations

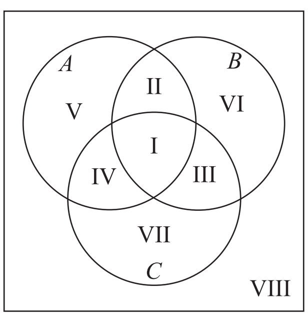


Figure 1.35

Sometimes the numbers do not arrange themselves in the Venn diagram as neatly as they did in the exercises in Section 2. If this happens you can often use deductive reasoning to fill in the diagram. However, sometimes a system of equations must be used to complete the diagram. If we use a system of linear equations to fill in the Venn diagram, we need a consistent way to refer to the eight regions of the Venn diagram. In Figure 1.35 we see the eight regions on the diagram labeled with the Roman numerals I through VIII. That is, $n(A \cap B^c \cap C^c)$ is V.

The next example will be solved using two different methods. If the solution is not unique, techniques from our study of linear systems can be used.

EXAMPLE 5 **Vegetable Survey** There were 32 students surveyed and asked if they did or did not like tomatoes, spinach, or peas with the results listed below. Arrange this information in a Venn diagram.

- 7 students liked all three vegetables
- 6 students did not like spinach or peas
- 1 student liked only tomatoes
- 9 students liked tomatoes and spinach

- 23 students liked two or more of these vegetables
- 5 students liked spinach but not peas
- 18 students liked spinach

Solution Begin with a blank Venn diagram. Let
$S = \{x|x \text{ is a student who likes spinach}\}$
$T = \{x|x \text{ is a student who likes tomatoes}\}$
$P = \{x|x \text{ is a student who likes peas}\}$

DEDUCTIVE REASONING METHOD:

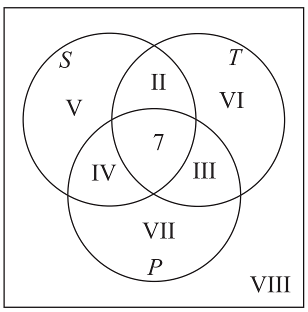


Figure 1.36

- The first clue that “7 students liked all three vegetables” tells us that we can place the number 7 in the intersection of all three sets: $n(S \cap T \cap P) = 7$. See Figure 1.36 You can scratch out this clue as each clue is used only once.
- The second clue that “6 students did not like spinach or peas” is not useful yet as both VI and VIII are outside of the region corresponding to liking spinach or peas. All we know is that those two numbers add to 6. Move on to the next clue.
- The third clue says “1 student liked only tomatoes” and so the number 1 is placed in region VI ($S^c \cap T \cap P^c$). Scratch out this clue as it has been used.

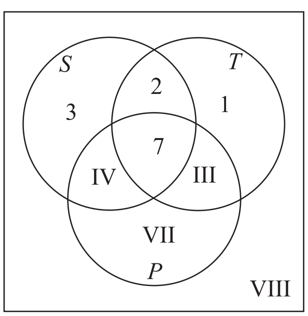


Figure 1.37

- The fourth clue says “9 students liked tomatoes and spinach” and this means 7 + II = 9. So there must be 2 students who like tomatoes and spinach but not peas. A 2 is placed in the region $S \cap T \cap P^c$ as shown in Figure 1.37. This clue can be scratched out.
- The fifth clue states that “23 students liked two or more of these vegetables.” The region representing two or more vegetables is shaded in Figure 1.37 and we see that is the sum of four numbers and at this point we only know the value of two of them. This clue is skipped for now.
- The sixth clue states “5 students liked spinach but not peas.” This is the region that is in the S circle but outside the P circle. We see that 2 of the 5 students have been accounted for as they like spinach and tomatoes but not peas. Therefore 3 students like only spinach and a 3 is placed in the region $S \cap T^c \cap P^c$ and this clue is scratched out.

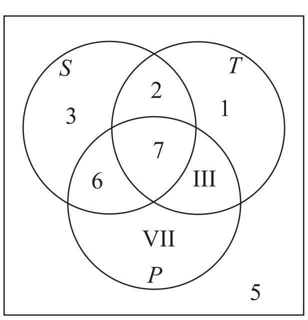


Figure 1.38

- The seventh clue is that “18 students liked spinach” and we see that the S circle is nearly complete. If there are 18 students in all in this circle and $3+2+7 = 12$ of them are accounted for, then $18-12 = 6$ students must be in the remaining empty spot in the S circle for students who like spinach and peas but not tomatoes, $S \cap T^c \cap P$. This is shown in Figure 1.38.
- The eighth clue is that 32 students are surveyed, so $n(U) = 32$. Since three numbers are still missing in the diagram, we are not ready for this clue yet.

We have three missing numbers in our diagram and three un-used clues:

- 6 students did not like spinach or peas

- 23 students liked two or more of these vegetables
- 32 students were surveyed

The region representing not spinach or peas will be those students who like only tomatoes (which is 1 student) and those students who do not like any of the vegetables, so $6 - 1 = 5$ students did not like any of the vegetables.

Referring back to the shaded region in Figure 1.37 we now know 3 of the four numbers that represent liking two or more of these vegetables so we can find the missing number by subtraction, $23 - 2 - 6 - 7 = 8$ students liked tomatoes and peas but not spinach. Place the 8 in the region $S^c \cap T \cap P$.

Finally we do not know how many students liked only peas, but we do know there are 32 students who were surveyed and so by subtraction, $32 - 3 - 2 - 1 - 6 - 7 - 8 - 5 = 0$, so a zero is placed in the region $S^c \cap T^c \cap P$. The completed diagram is shown in Figure 1.39.

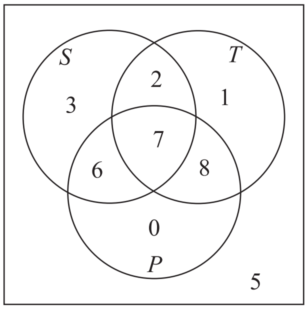


Figure 1.39

SYSTEMS OF EQUATIONS METHOD:

Referring to Figure 1.40 with the Roman numerals, we translate each clue into a linear equation.

- 32 students were surveyed → I + II + III + IV + V + VI + VII + VIII = 32
- 7 students liked all three vegetables → V = 7
- 6 students did not like spinach or peas → III + VIII = 6
- 1 student liked only tomatoes → III = 1
- 9 students liked tomatoes and spinach → II + V = 9
- 23 students liked two or more of these vegetables → II + IV + V + VI = 23
- 5 students liked spinach but not peas → I + II = 5
- 18 students liked spinach → I + II + IV + V = 18

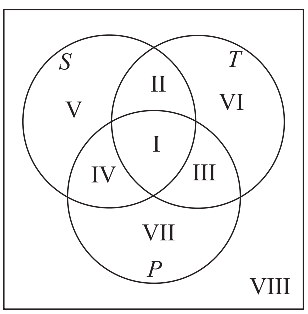


Figure 1.40

Place this information into an augmented matrix and use the methods of Chapter 1 to solve the system. This method is particularly useful if a calculator is used. With $x_1 = \text{I}$, $x_2 = \text{II}$, and so on we have

$$\begin{array}{l}
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 32 \\
x_5 = 7 \\
x_3 + x_8 = 6 \\
x_3 = 1 \\
x_2 + x_5 = 9 \\
x_2 + x_4 + x_5 + x_6 = 5 \\
x_1 + x_2 = 32 \\
x_1 + x_2 + x_4 + x_5 = 18
\end{array}$$

The solution to the system is

I = 3, II = 2, III = 1, IV = 6, V = 7, VI = 8, VII = 0, VIII = 5. ✦

Complete a Venn diagram for each of the exercises below.

1. $n(A) = 11, n(B) = 14, n(C) = 19, n(U) = 36, n(A \cap B) = 4, n(A \cap C) = 8, n(B \cap C) = 7$ and $n(A \cup B \cup C) = 28$

2. A class of 6th grade boys is surveyed and asked if they were wearing one or more of the following items of clothing that day: a T-shirt, shorts or athletic shoes. The results were:

- 40 wore a T-shirt, shorts, and athletic shoes
- 23 wore exactly two of these items
- 57 wore shorts
- 12 did not wear a T-shirt
- 4 wore only athletic shoes
- 7 did not wear shorts or athletic shoes
- 11 wore only shorts and a T-shirt
- 52 wore a T-shirt and athletic shoes

3. Two hundred tennis players were asked which of these strokes they considered their weakest stroke(s): the serve, the backhand, and the forehand.

- 20 players said none of these were their weakest stroke
- 30 players said all three of these were their weakest stroke
- 40 players said their serve and forehand were their weakest strokes
- 40 players said that only their serve and backhand were their weakest strokes.
- 15 players said that their forehand but not their backhand was their weakest stroke
- 52 players said that only their backhand was their weakest stroke
- 115 players said their serve was their weakest stroke

Source: Joe Kahlig

4. Thirty-one children were asked about their lunch preferences and the following results were found:

- 12 liked cheeseburgers
- 14 liked pizza
- 9 liked burritos
- 5 liked cheeseburgers and pizza
- 4 liked cheeseburgers and burritos
- 8 liked pizza and burritos
- 10 liked none of these items

Answers to Selected Exercises

L.1 EXERCISES

1. Statement **3.** Statement **5.** Statement
7. Not a statement **9.** Statement
11. Not a statement **13.** Statement
15. **a.** George Washington was not the third president of the United States. **b.** George Washington was the third president of the United States, and Austin is the capital of Texas. **c.** George Washington was the third president of the United States, or Austin is the capital of Texas. **d.** George Washington was not the third president of the United States, and Austin is the capital of Texas. **e.** George Washington was the third president of the United States, or Austin is not the capital of Texas. **f.** It is not true that George Washington was the third president of the United States and that Austin is the capital of Texas.
17. **a.** George Washington did not own over 100,000 acres of property. The Exxon Valdez was not a luxury liner. **b.** George Washington owned over 100,000 acres of property, or the Exxon Valdez was a luxury liner.
c. George Washington owned over 100,000 acres of property, and the Exxon Valdez was a luxury liner.
19. **a.** $\sim q$ **b.** $p \wedge \sim q$ **c.** $p \vee q$ **d.** $(\sim p) \vee (\sim q)$

L.2 EXERCISES

1.

p	q	$\sim q$	$p \wedge \sim q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

3.

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

5.

p	q	$\sim q$	$p \wedge \sim q$	$(p \wedge \sim q) \vee q$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	F

7.

p	q	$\sim p$	$p \wedge q$	$\sim p \vee (p \wedge q)$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

9.

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \wedge (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	F

11.

p	q	$\sim q$	$\sim p$	$p \vee \sim q$	$\sim p \wedge q$	$(p \vee \sim q) \vee (\sim p \wedge q)$
T	T	F	F	T	F	T
T	F	T	F	T	F	T
F	T	F	T	F	T	T
F	F	T	T	T	F	T

13.

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

15.

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$\sim[(p \wedge q) \wedge r]$
T	T	T	T	T	F
T	T	F	T	F	T
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	F	F	T
F	F	F	F	F	T

17.

p	q	r	$p \vee q$	$q \wedge r$	$(p \vee q) \vee (q \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	F	F	F

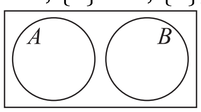
19.

p	q	r	$\sim q$	$p \vee \sim q$	$\sim q \wedge r$	$(p \vee \sim q) \vee (\sim q \wedge r)$
T	T	T	F	T	F	T
T	T	F	F	T	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	T	T	T	T
F	F	F	T	T	F	T

21. **a.** True **b.** False **c.** True **d.** True **e.** False
23. **a.** True **b.** False **c.** False **d.** True

1.1 EXERCISES

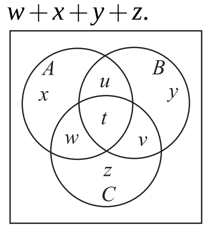
1. **a.** False **b.** False **3.** **a.** True **b.** False
5. **a.** False **b.** True **c.** False (d) True
7. **a.** $\emptyset$, $\{3\}$ **b.** $\emptyset$, $\{3\}$, $\{4\}$, $\{3,4\}$

9. 

11. **a.** 2 **b.** 3 disjoint
13. **a.** 1, 2, 3 **b.** 1 not disjoint
15. **a.** I **b.** V
17. **a.** VIII **b.** IV
19. **a.** II, V, VI **b.** III, IV, VII
21. **a.** VII **b.** I, II, III, IV, V, VII, VIII
23. **a.** $\{4,5,6\}$ **b.** $\{1,2,3,4,5,6,7,8\}$
25. **a.** $\{1,2,3\}$ **b.** $\{9,10\}$
27. **a.** $\{5,6\}$ **b.** $\{1,2,3,4,7,8,9,10\}$
29. **a.** $\emptyset$ **b.** $\emptyset$
31. **a.** People in your state who do not own an automobile
b. People in your state who own an automobile or house
c. People in your state who own an automobile or not a house
33. **a.** People in your state who own an automobile but not a house
b. People in your state who do not own an automobile and do not own a house
c. People in your state who do not own an automobile or do not own a house
35. **a.** People in your state who own an automobile and a house and a piano
b. People in your state who own an automobile or a house or a piano
c. People in your state who own both an automobile and a house or else own a piano
37. **a.** People in your state who do not own both an automobile and a house but do own a piano
b. People in your state who do not own an automobile, nor a house, nor a piano
c. People in your state who own a piano, but do not own a car or a house
39. **a.** $N \cap F$ **b.** $N \cap H^c$ **41.** **a.** $N \cup S$ **b.** $N^c \cap S^c$
43. **a.** $(N \cap H) \cup (S \cap H)$ **b.** $(N \cap F) \cap H^c$
45. **a.** $(F \cap H) \cap (N \cup S)^c$ **b.** $F \cap H^c \cap N^c \cap S^c$
47. Both expressions give U
49. Both expressions give $\{1,2,3,4,5,6,7\}$
51. Both expressions give $\{8,9,10\}$.

1.2 EXERCISES

1. 135 **3.** 70 **5.** 60 **7.** 150 **9.** 110
11. 90 **13.** 15 **15.** 7 **17.** 3 **19.** 56
21. **a.** 1100 **b.** 750 **c.** 100
23. **a.** 100 **b.** 500
25. **a.** 30 **b.** 360 **c.** 70
27. **a.** 20 **b.** 25 **c.** 345 **29.** 30
31. From the figure we have $n(A) - n(A \cap B) = x + w$, $n(B) - n(B \cap C) = u + y$, $n(C) - n(A \cap C) = v + z$, and $n(A \cap B \cap C) = t$. Adding these four equations gives the result since from the figure $n(A \cup B \cup C) = t + u + v + w + x + y + z$.



1.3 EXERCISES

1. $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$
3. $S = \{(H,H,H), (H,H,T), (H,T,H), (T,H,H), (H,T,T), (T,H,T), (T,T,H), (T,T,T)\}$, $E = \{(H,H,H), (H,H,T), (H,T,H), (T,H,H)\}$
5. $S = \{(1,1,1), (1,1,0), (1,0,1), (0,1,1), (0,0,1), (0,1,0), (1,0,0), (0,0,0)\}$, $E = \{(1,1,1), (1,1,0), (1,0,1), (0,1,1)\}$
7. **a.** $\{\text{black}, \text{white}, \text{red}\}$ **b.** $\{\text{black}, \text{red}\}$
9. $\{(A,B,C), (A,B,D), (A,B,E), (A,C,D), (A,C,E), (A,D,E), (B,C,D), (B,C,E), (B,D,E), (C,D,E)\}$, $\{(A,B,C), (A,B,D), (A,B,E), (B,C,D), (B,C,E), (B,D,E)\}$
11. $S = \{1,2,3,4,6,9\}$, $E = \{2,4,6\}$
13. **a.** $\{0,1,2,3,4,5,6,7,8,9,10\}$ **b.** $E = \{6,7,8,9,10\}$ **c.** $F = \{0,1,2,3,4\}$
d. $E \cup F = \{0,1,2,3,4,6,7,8,9,10\}$. $E \cap F = \emptyset$. $E^c = \{0,1,2,3,4,5\}$. $E \cap F^c = \{6,7,8,9,10\}$. $E^c \cap F^c = \{5\}$
e. E^c and $E \cap F^c$. $E \cap F^c$ and $E^c \cap F^c$. $E \cup F$ and $E^c \cap F^c$.
15. **a.** $\{(S,S,S), (S,S,F), (S,F,S), (F,S,S), (S,F,F), (F,S,F), (F,F,S), (F,F,F)\}$
b. $E = \{(S,S,S), (S,S,F), (S,F,S), (F,S,S)\}$
c. $G = \{(S,S,S), (S,S,F), (S,F,S), (S,F,F)\}$
d. $E \cup G = \{(S,S,S), (S,S,F), (S,F,S), (S,F,F), (F,S,S)\}$, $E \cap G = \{(S,S,S), (S,S,F), (S,F,S)\}$, $G^c = \{(F,S,S), (F,S,F), (F,F,S), (F,F,F)\}$, $E^c \cap G = \{(S,F,F)\}$, $(E \cup G)^c = \{(F,F,S), (F,S,F), (F,F,F)\}$.
e. $E \cap G$ and G^c. $E \cap G$ and $E^c \cap G$. $E \cap G$ and $(E \cup G)^c$. G^c and $E^c \cap G$. $E^c \cap G$ and $(E \cup G)^c$. $E \cup G$ and $(E \cup G)^c$.
17. **a.** Pencils that are longer than 10 cm and less than 25 cm.
b. Pencils that are less than 10 cm long.
c. Pencils that are 25 cm or longer. **d.** $\emptyset$
19. $E \cap F^c$ **21.** $F^c \cap E^c$ **23.** $E \cap F \cap G^c$
25. $E \cup F \cup G =$ all 26 letters. $E^c \cap F^c \cap G^c = \emptyset$. $E \cap F \cap G = \emptyset$. $E \cup F^c \cup G = \{a,e,i,o,u,b,c,d\}$
27. $S = \{r,b,g,y\}$, $E = \{b,g\}$

1.4 EXERCISES

1. $1/2$ **3.** $2/3$ **5.** $1/13$ **7.** $1/2$
9. $3/26$ **11.** $1/4$ **13.** $2/3$ **15.** 0.15
17. a. 0.12 **b.** 0.8 **c.** 0.08 **19.** 0.56
21. A: 0.125, B: 0.175, C: 0.4, D: 0.2, F: 0.1
23. 0.55 **25.** $1/2$ **27.** $1/2$ **29.** $3/8$, $7/8$
31. $1/12$ **33.** $1/36$, $1/12$, $5/36$ **35.** $1/3$

1.5 EXERCISES

1. 0.50, 0.90 **3.** 0.18, 0.63, 0.62 **5.** 0.25
7. $1/6$ **9.** $7/30$ **11.** 0.60, 0.80, 0
13. 0.70, 0.70, 0, 0.30, 1 **15.** 0.60, 0.10
17. 0.20, 0.10 **19.** 0.20, 0.10, 0.15
21. 0.85 **23. a.** 3:17 **b.** 1:3
25. 2:3 **27.** 7:5 **29.** 0.20
31. 0.04, 0.96 **33.** 0.039, 0.049, 0.951
35. a. 0.009 **b.** 0.001 **c.** 0.006 **d.** 0.974
37. 0.20
39. You do not know what the actual probability is. You do know that the empirical probability is $165/1000 = 0.165$. This represents the best guess for the actual probability. But if you tossed the coin more times, the relative frequency and the new empirical probability would most likely had changed.
41. The probabilities in the game are constant and do not change just because you are on a winning streak. Thus no matter what has happened to you in the past, the probability of winning any one game remains constant at 0.48. Thus if you continue to play, you should expect to win 48% of the time in the future. You have been lucky to have won 60% of the time up until now.
43. After reading the first discussion problem above, we know that it is, in fact, impossible to determine with certainty the actual probability precisely. Since the die has been tossed a total of 2000 times and a one has come up 335 times, our best guess at the probability is $335/2000 = 0.1675$.

1.6 EXERCISES

1. $3/7$, $3/5$ **3.** $2/3$, $4/5$ **5.** 1, 0
7. $2/3$, $1/2$ **9.** $1/3$, 1 **11.** 0, $1/2$
13. No **15.** Yes **17.** No
19. Yes **21.** $2/11$ **23.** $1/7$
25. a. $\frac{10{,}200}{132{,}600} \approx 0.077$ **b.** $\frac{49}{25\cdot 33} \approx 0.059$
27. 0.12, 0.64, 0.60 **29.** $1/3$, $3/10$
31. No **33.** 0.65 **35.** 0.72
37. 0.02, 0.017 **39.** 0.026
41. 0.000001 **43.** $5/7$, $5/21$, $1/21$
45. Yes **47.** No **49.** 0.057818
51. For E and F to be independent, they must satisfy $P(E) \times P(F) = P(E \cap F)$. From the Venn diagram, we must have: $(p_1 + p_2) \times (p_3 + p_2) = p_2$. So,

$$p_1 p_3 + p_2 p_3 + p_1 p_2 + p_2^2 = p_2$$

$$p_1 p_3 = p_2(1 - p_3 - p_2 - p_1) = p_2 p_4$$

The above steps can be reversed, so if $p_1 p_3 = p_2 p_4$, we will have $P(E) \times P(F) = P(E \cap F)$.

If the sets are mutually disjoint, then $p_2 = 0$. This implies that $p_1 p_3 = p_2 p_4 = 0$. Then either p_1 or p_3 or both are zero. Thus either $P(E) = 0$, $P(F) = 0$, or both.
53. Since E and F are independent, $P(E) \times P(F) = P(E \cap F)$.

$$\begin{aligned} P(E^c \cap F^c) &= 1 - P(E \cup F) \\ &= 1 - \big(P(E) + P(F) - P(E \cap F)\big) \\ &= 1 - P(E) - P(F) + P(E) \times P(F) \\ &= \big(1 - P(E)\big) \times \big(1 - P(F)\big) \\ &= P(E^c) \times P(F^c) \end{aligned}$$

Hence, if E and F are independent, so are E^c and F^c.
55. Since E and F are exclusive, $P(E \cap F) = P(\emptyset) = 0$. Since $P(E)$ and $P(F)$ are both nonzero, then $P(E) \times P(F) > 0$. Therefore, E and F are not independent.
57.

$$\begin{aligned} P(E^c | F) &= \frac{P(E^c \cap F)}{P(F)} \\ &= \frac{P(F) - P(E \cap F)}{P(F)} \\ &= 1 - P(E|F) \end{aligned}$$

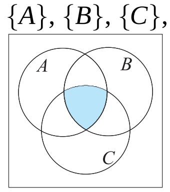
59. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$
61. $P(E|F) + P(E^c|F) = \frac{P(E \cap F) + P(E^c \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$

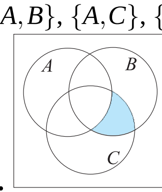
1.7 EXERCISES

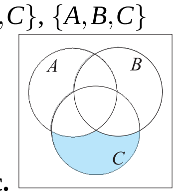
1. $3/7$, $57/93$ **3.** $3/19$, $1/3$ **5.** $4/11$
7. a. $2/23$ **b.** $21/23$ **9.** $2/17$
11. a. $1/56$ **b.** $5/28$ **13.** $2/7$
15. $4/19$ **17.** $13/51$ **19.** $13/51$
21. $6/17$ **23.** $100/136 \approx 0.74$
25. $396/937 \approx 0.42$, $5/937 \approx 0.005$
27. $5/7$ **29.** $10/19$ **31. a.** $1/12$ **b.** $1/4$
33. $P(1|N) = 6/20$, $P(2|N) = 4/20$, $P(3|N) = 6/20$, $P(4|N) = 4/20$

REVIEW EXERCISES

1. a. Yes **b.** No **c.** Yes
2. $\{x \mid x = 5n, n$ is an integer and $1 \le n \le 8\}$
3. $\{0, -\sqrt{2}, \sqrt{2}\}$
4. $\emptyset$, $\{A\}$, $\{B\}$, $\{C\}$, $\{A,B\}$, $\{A,C\}$, $\{B,C\}$, $\{A,B,C\}$

5. a.


b.


c.


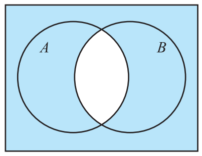
6. $A \cup B = \{1,2,3,4\}$, $A \cap B = \{2,3\}$, $B^c = \{1,5,6\}$, $A \cap B \cap C = \{2,3\} \cap \{4,5\} = \emptyset$, $(A \cup B) \cap C = \{4\}$, $A \cap B^c \cap C = \emptyset$
7. a. My current instructors who are less than 6 feet tall. **b.** My current instructors who are at least 6 feet tall or are male. **c.** My current instructors who are female and weigh at most 180 pounds. **d.** My current male instructors who are at least 6 feet tall and weigh more than 180 pounds. **e.** My current male instructors who are less than 6 feet tall and weigh more

than 180 pounds. **f.** My current female instructors who are at least 6 feet tall and weigh more than 180 pounds.

8. **a.** M^c **b.** $M^c \cap W^c$ **c.** $M \cap (H \cup W)$

9. $\{1,2,3,4\} = \{1,2,3,4\}$

10. The shaded area is $(A \cap B)^c = A^c \cup B^c$.



11. 120 **12.** 10 **13.** 50

14. **a.** 40 **b.** 45 **c.** 19

15. **a.** 0.016, 0.248, 0.628, 0.088, 0.016, 0.044 **b.** 0.892 **c.** 0.264

16. **a.** $5/30$ **b.** $15/30$ **c.** $20/30$

17. $P(E \cup F) = 0.60$, $P(E \cap F) = 0$, $P(E^c) = 0.75$

18. $P(E \cup F) = 0.55$, $P(E^c \cap F) = 0.35$, $P((E \cup F)^c) = 0.45$

19. $P(B) = 1/6$ **20.** 0.75

21. **a.** 0.09 **b.** 0.91

22. **a.** 0.13 **b.** 0.09 **c.** 0.55

23. $P(E|F) = 0.40$, $P(E^c|F) = 0.60$, $P(F^c|E^c) = 4/7$

24. Not independent

25. $(0.05)^3$

26. 0.254

27. **a.** $2/31$ **b.** $4/31$

28. 0.50

2.1 EXERCISES

1. $5 \times 4 \times 3 = 60$ **3.** $8 \times 7 \times 6 \times 5 \times 4 = 6720$

5. $7! = 5040$ **7.** $9! = 362{,}880$ **9.** $9 \times 8 = 72$

11. n **13.** $4 \times 30 = 120$ **15.** $4 \times 10 \times 5 = 200$

17. $10^6 = 1{,}000{,}000$ **19.** $2 \times 10 \times 5 \times 3 = 300$

21. $(26)^3(10)^3 = 17{,}576{,}000$ **23.** 96

25. $5 \times 4 \times 4 \times 3 \times 3 \times 2 \times 2 = 2880$ **27.** $5! = 120$

29. $2!4!6! = 34{,}560$ **31.** $9!5! = 43{,}545{,}600$

33. $3!4! = 144$ **35.** $12 \times 11 \times 10 \times 9 = 11{,}880$

37. 1,814,400 **39.** $(10 \times 9 \times 8)(8 \times 7 \times 6) = 241{,}920$

41. $P(11,4) \times P(9,3) \times P(5,2) = 79{,}833{,}600$

43.
$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= n(n-1)(n-2)\cdots(n-r+1)\frac{(n-r)(n-r-1)\cdots 2\cdot 1}{(n-r)(n-r-1)\cdots 2\cdot 1} \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)(n-r-1)\cdots 2\cdot 1}{(n-r)(n-r-1)\cdots 2\cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

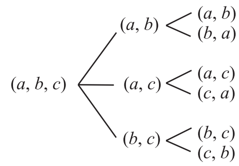
45. 24

2.2 EXERCISES

1. $\frac{8\times7\times6}{3\times2\times1} = 56$ **3.** $\frac{8\times7\times6\times5\times4}{5\times4\times3\times2\times1} = 56$ **5.** 12

7. 35 **9.** 105 **11.** n

13.



15. 10

17. $C(46,6) = \frac{46!}{40!\times 6!} = 9{,}366{,}819$

19. $C(20,5) = \frac{20!}{15!\times 5!} = 15{,}504$

21. $C(20,4) = \frac{20!}{16!\times 4!} = 4845$

23. $C(40,5) = \frac{40!}{5!\times 35!} = 658{,}008$

25. $C(11,3) \times C(7,2) = 3465$

27. $C(20,7) \times C(6,3) = 1{,}550{,}400$

29. $C(12,3)P(9,3) = 110{,}880$

31. $C(11,4)P(7,2) = 13{,}860$

33. $C(9,5) = 126$

35. $C(8,5) = 56$

37. $2^8 - C(8,0) - C(8,1) = 256 - 1 - 8 = 247$

39. $C(12,3)C(8,3)C(4,2)2 = 147{,}840$

41. $C(4,3) \cdot 48 \cdot 44/2 = 4224$

43. $C(4,2)C(4,2)C(13,2) \cdot 44 = 123{,}552$

45.
$$\begin{aligned} C(n,r) &= \frac{n!}{(r!)(n-r)!} \\ &= \frac{n!}{(n-r)!(n-(n-r))!} \\ &= C(n, n-r) \end{aligned}$$

2.3 EXERCISES

1. $3/10$, $1/3$ **3.** $(6\cdot7\cdot8)/1330$ **5.** $20/323$ **7.** $6/1326$

9. $78/1326$ **11.** $208/1326$ **13.** $4/2{,}589{,}960$

15. $624/2{,}589{,}960$ **17.** $10{,}200/2{,}589{,}960$

19. $123{,}552/2{,}589{,}960$

21. 0.0833; 0.2361; 0.4271; 0.6181

23. 1 **25.** $6/45$ **27.** $56/2^{10} \approx 0.0547$

29. $5/11$ **31.** 420 **33.** 34,650 **35.** 2520

37. **a.** 6 **b.** 3 **c.** 24 **d.** 12 **e.** 6 **39.** 3780

2.4 EXERCISES

1. $5(.2)^4(.8) = .0064$ **3.** $35(.5)^7 \approx 0.273$

5. $70(.25)^4(.75)^4 \approx 0.087$ **7.** $20(.5)^6 \approx 0.3125$

9. $35(.1)^4(.9)^3 \approx 0.00255$ **11.** $10(.1)^3(.9)^2 \approx 0.0081$

13. $45 \times (0.5)^{10} \approx 0.044$ **15.** $56(0.5)^{10} \approx 0.0547$

17. $11(0.5)^{10} \approx 0.0107$ **19.** $7(0.6)^6(0.4) \approx 0.1306$

21. $7(0.6)^6(0.4) + (0.6)^7 \approx 0.1586$

23. $(0.4)^7 + 7(0.6)(0.4)^6 + 21(0.6)^2(0.4)^5 \approx 0.0962$

25. $1 - (0.633)^4 \approx 0.839$ **27.** $(0.839)^{10} \approx 0.173$

29. $1 - (0.915)^4 - 4(0.085)(0.915)^3 \approx 0.03859$

31. $(0.0386)^3 \approx 0.0000575$

33. $12(0.05)(0.95)^{11} \approx 0.341$

35. $(.95)^{12} + 12(.05)(.95)^{11} + 66(.05)^2(.95)^{10} \approx 0.980$

37. $C(20,10)(0.8)^{10}(0.2)^{10} \approx 0.002$

39. $190(.8)^{18}(.2)^2 + 20(.8)^{19}(.2) + (.8)^{20} \approx 0.206$

41. $190(0.05)^2(0.95)^{18} \approx 0.189$ **43.** ≈ 0.984

2.5 EXERCISES

1. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

3. $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$

5. $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$