3.4 The Chain Rule

The methods we have seen so far do not tell us how to differentiate for example the function \( F(x) = \sqrt{x^2 + 1} \). Note that \( F \) is the composition of two functions \( f(x) = \sqrt{x} \) and \( g(x) = x^2 + 1 \). The chain rule enables us to differentiate such functions:

**The Chain Rule:** If \( g \) is differentiable at \( x \) and \( f \) is differentiable at \( g(x) \), then the composite function \( F = f \circ g \) defined by \( F(x) = f(g(x)) \) is differentiable at \( x \) and \( F' \) is given by

\[
F'(x) = f'(g(x))g'(x)
\]

In other notation, if \( y = f(u) \) and \( u = g(x) \) are both differentiable functions then

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}
\]

**Activity 1:** For each function given below, identify an inner function \( u \) and outer function \( f \) to write the function in the form \( f(u(x)) \). Then, determine \( f'(x) \), \( g'(x) \), and \( f'(g(x)) \), and finally apply the chain rule to determine the derivative of the given function.

(a) \( F(x) = \sqrt{x^2 + 1} \)
(b) \( y = \sin(x^2) \)

(c) \( y = \sin^2(x) \)

(d) \( f(x) = (1 + x^5)^{4/5} \)
(e) \( f(x) = e^x \cos x \)

(f) \( f(x) = \left( \frac{x^2 - 1}{x + 2} \right)^5 \)

(g) \( f(t) = \sqrt{\frac{t}{t^2 + 4}} \)
3.7 Derivatives of Logarithmic Functions

**Preview Activity:** The below figure shows the graph of \( f(x) = \ln x \) and its tangent lines at the points \( x = 1/2, 1, 3, \) and \( 1/5. \)

![Graph of \( f(x) = \ln x \)](image)

(a) By looking at the above figure, order the slopes of the tangent lines at the given points.

(b) Does the graph of \( f(x) = \ln x \) have any horizontal tangent lines?

(c) Based on your work in (a) and (b), sketch an estimated graph of \( y = f'(x). \)

(d) What familiar function do you think is the derivative of \( f(x) = \ln(x) \)?
**Activity 1:** Use the derivative of \( \ln x \) and change of base formula to find \( \frac{d}{dx}(\log_a x) \).

**Activity 2:** Use the derivative of \( \ln x \) and the chain rule to find \( \frac{d}{dx} \ln(f(x)) \).

**Activity 3:** Differentiate

(a) \( y = \ln(2\sqrt{x}) \)
(b) \( y = \ln(3 \sin^2(x)) \)

(c) \( y = (\cos x)(\ln x) \)

(d) \( y = [\ln(1 + e^x)]^2 \)