4. APPLICATIONS OF DIFFERENTIATION

In this chapter we will study various applications of the derivative. For example, we will use methods of calculus to analyze functions and their graphs. Another important application of the derivative will be in the solution of optimization problems. For example, if time is the main consideration in a problem, we might be interested in finding the quickest way to perform a task, and if cost is the main consideration, we might be interested in finding the least expensive way to perform a task. Mathematically, optimization problems can be reduced to finding the largest or smallest value of a function on some interval, and determining where the largest or smallest value occurs. Using the derivative, we will develop the mathematical tools necessary for solving such problems.

4.2 Maximum and Minimum Values

**Definition:** The number $f(a)$ is:

- a **local maximum** value of $f$ if ..$f(a) \geq f(x)$...... when $x$ is near $a$.
- a **local minimum** value of $f$ if ..$f(a) \leq f(x)$...... when $x$ is near $a$.
- an **absolute maximum** value of $f$ if ..$f(a) \geq f(x)$...... for all $x$ in the domain of $f$.
- an **absolute minimum** value of $f$ if ..$f(a) \leq f(x)$...... for all $x$ in the domain of $f$.
- The maximum and minimum values of a function $f$ are called the **extreme** values of $f$. 
Activity 1: Consider the function \( h \) whose graph is given below. Use the graph to answer each of the following questions.

(a) Identify all of the values of \( a \) for which \( h(a) \) is a local maximum of \( h \).
\[
 a = -2 \quad , \quad a = 1
\]
(b) Identify all of the values of \( a \) for which \( h(a) \) is a local minimum of \( h \).
\[
 a = 0
\]
(c) Does \( h \) have a absolute maximum? If so, what is the value of this absolute maximum?
\[
 a = 1 \quad , \quad f(1) = 2
\]
(d) Does \( h \) have a absolute minimum? If so, what is its value?
\[
 a = -3 \quad , \quad f(-3) = -3
\]
(e) Identify all values of \( a \) for which \( h'(a) = 0 \).
\[
 a = 1 \quad , \quad a = 2.5
\]
(f) Identify all values of \( a \) for which \( h'(a) \) does not exist.
\[
 a = -2 \quad , \quad a = 0 \quad , \quad a = 1.5
\]
(g) True or false: every relative maximum and minimum of \( h \) occurs at a point where \( h'(a) \) is either zero or does not exist.
\[
 \begin{align*}
 \text{max} & \quad \frac{\text{min}}{\text{a}=0} \quad \frac{\text{min}}{\text{a}=1} \\
 h'(a) & \text{ is DNE} \quad h'(a) = 0
\end{align*}
\]
(h) True or false: at every point where \( h'(a) \) is zero or does not exist, \( h \) has a relative maximum or minimum.
\[
 a = -2.5 \quad , \quad \text{a does not exist, } a = 1.5
\]

Definition: A critical number (or point) of a function \( f \) is a number \( a \) in the domain of \( f \) such that either \( f'(a) = 0 \) or \( f'(a) \) does not exist.
Activity 2: Find the critical numbers of the following functions.

(a) \( f(x) = x^{3/5}(4 - x) = 4x^{3/5} - x^{8/5} \)

\[
\frac{f'(x)}{x^{2/5}} = \frac{3}{5}x^{-2/5} - \frac{8}{5}x^{3/5} = \frac{1}{5}x^{-2/5}(12 - 8x) = \frac{12 - 8x}{x^{2/5}}
\]

\( f'(x) = 0 \iff 12 - 8x = 0 \)

\[
\begin{aligned}
x &= \frac{12}{8} - \frac{3}{2} \\
\text{Dom} (f') &= (-\infty, 0) \cup (0, \infty) \\
f'(0) &\text{ does not exist.}
\end{aligned}
\]

(b) \( f(x) = |x - 3| \)

at \( x = 3 \) \( f'(3) \) does not exist.

\[
\begin{aligned}
f(x) &= \begin{cases} 
  x - 3 & \text{if } x \geq 3 \\
  -(x - 3) & \text{if } x < 3 
\end{cases} \\
f'(x) &= \begin{cases} 
  1 & \text{if } x > 3 \\
  -1 & \text{if } x < 3 
\end{cases}
\end{aligned}
\]

\( \text{Dom} (f') = (-\infty, 2) \cup (3, \infty) \)

The only critical number of \( f \) is \( x = 3 \).

(c) \( f(x) = \frac{(x + 1)^2}{x - 1} \)

\[
\begin{aligned}
f'(x) &= \frac{2(x+1)(x-1) - (x+1)^2}{(x-1)^2} \\
&= \frac{(x+1)(2x-2 - x-1)}{(x-1)^2} \\
&= \frac{(x+1)(x-3)}{(x-1)^2}
\end{aligned}
\]

\( f'(x) = 0 \iff x = -1 \text{ or } x = 3 \)

\( f'(3) \text{ is DNE} \)

Critical numbers of \( f \): \( x = -1, x = 1, x = 3 \)
Activity 3: Give an example of a function with

- infinitely many extreme values

- a minimum value but no maximum value

- no maximum or minimum

- extreme values attained at an endpoint

The next two theorems tell us when and where we always have extreme values.

The Extreme Value Theorem: If \( f \) is a continuous function defined on a closed interval \([a, b]\), then it has an absolute maximum value \( f(c) \) and an absolute minimum value \( f(d) \) for some \( c \) and \( d \) in \([a, b]\).

Theorem: If \( f \) is a continuous function defined on a closed interval \([a, b]\), then its absolute maximum or minimum can only occur at a critical number \( c \) in \((a, b)\) or at the end points \( a \) or \( b \).
The Closed Interval Method: To find the absolute maximum and minimum values of a continuous function \( f \) on a closed interval \([a, b]\):

- **Step 1.** Find the critical numbers of \( f \) in \((a, b)\) and find the values of \( f \) at the critical numbers.
- **Step 2.** Find the values of \( f \) at the endpoints \( a \) and \( b \).
- **Step 3.** The largest of the values from Step 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

**Activity 4:** Find the absolute maximum and absolute minimum values of the following functions on the given intervals.

(a) \( f(x) = x^3 - 6x^2 + 9x + 2 \) on the interval \([-1, 4]\).

\[
\begin{align*}
    f'(x) &= 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3) = 0 \\
    \Rightarrow x &= 1 \text{ or } x = 3 \\
    f(1) &= 1 - 6 + 9 + 2 = 6 \\
    f(3) &= 3^3 - 6.3^2 + 9.3 + 2 = 2 \\
    f(-1) &= -1 - 6 - 9 + 2 = -14 \\
    f(4) &= 4^3 - 6.4^2 + 9.4 + 2 = 6 \\
\end{align*}
\]

**Critical numbers:**

- Absolute max. at \( x = 1, x = 4 \)
- Absolute min. at \( x = -1 \)

(b) \( f(t) = t\sqrt{4-t^2} \) on the interval \([-1, 2]\).

\[
\begin{align*}
    f'(t) &= \frac{\sqrt{4-t^2} + t \cdot \frac{-2t}{2\sqrt{4-t^2}}}{4-t^2} = \frac{4-2t^2 - t^2}{\sqrt{4-t^2}} = \frac{4-2t^2}{\sqrt{4-t^2}} = 0 \\
    \Rightarrow 4-2t^2 &= 0 \\
    t^2 &= 2 \\
    t &= \sqrt{2} \text{ or } t = -\sqrt{2} \\
\end{align*}
\]

**Critical numbers:**

- \( t = \sqrt{2} \)
- \( t = -\sqrt{2} \)

**End pts.:** \( t = -1 \)

(c) \( f(x) = x - \ln x \) on the interval \([1/2, 2]\).

\[
\begin{align*}
    f'(x) &= 1 - \frac{1}{x} = \frac{x-1}{x} = 0 \\
    \Rightarrow x &= 1 \\
    f(1/2) &\notin [\text{Dom } f'] \\
    f(1) &= 1 \\
    f(2) &= 2 - \ln 2 = 1.19 \\
\end{align*}
\]

**Absolute max:** \( f(\sqrt{2}) = \sqrt{2} \sqrt{4-(\sqrt{2})^2} = 2 \)

**Absolute min:** \( f(-1) = -1 \sqrt{4-(-1)^2} = -2 \)

\( \odot 2015 \) Fatma Terzioglu
4.3 Derivatives and the Shape of Curves

In section 2.8 we have discussed how the signs of the first and second derivatives of a function shape its graph. Let us recall what we have learned.

- If \( f'(x) > 0 \) on an interval, then \( f \) is \( \text{increasing} \) on that interval.
- If \( f'(x) < 0 \) on an interval, then \( f \) is \( \text{decreasing} \) on that interval.
- If \( f''(x) > 0 \) on an interval, then \( f \) is \( \text{concave up} \) on that interval.
- If \( f''(x) < 0 \) on an interval, then \( f \) is \( \text{concave down} \) on that interval.

We also learned how to identify a local maximum or minimum by looking at the first derivative:

If \( c \) is a critical number of a continuous function \( f \),

- If \( f' \) changes its sign from positive to negative at \( c \), then \( f \) has a local \( \text{maximum} \) at \( c \).
- If \( f' \) changes its sign from negative to positive at \( c \), then \( f \) has a local \( \text{minimum} \) at \( c \).

We can also identify a local maximum or minimum by looking at the second derivative:

**The Second Derivative Test:** Suppose \( f'' \) is continuous near \( c \) and \( f'(c) = 0 \),

- If \( f''(c) > 0 \) then \( f \) has a local \( \text{minimum} \) at \( c \).
- If \( f''(c) < 0 \) then \( f \) has a local \( \text{maximum} \) at \( c \).
- If \( f''(c) = 0 \) then the test is \( \text{inconclusive} \).
General Graphing Strategy:

Step 1. Analyze \( f(x) \) by finding its domain and all asymptotes.

Step 2. Analyze \( f'(x) \): Create sign chart for \( f'(x) \) and determine intervals of increasing/decreasing and local extrema.

Step 3. Analyze \( f''(x) \): Create sign chart for \( f''(x) \) and determine intervals of concave up, concave down, and inflection points.

Step 4. Sketch the graph of \( f \) using all of the above information. Plot additional points as needed.

Activity 1: Use the graphing strategy to sketch a graph of each of the functions below:

(a) \( f(x) = \frac{-1}{4}(x+1)(x-2)^2 \)

Step 1. \( \text{Dom}(f) = (-\infty, \infty) \)

No asymptotes.

Step 2. \( f'(x) = \frac{-1}{4} \left[ (x-2)^2 + (x+1) \cdot 2(x-2) \right] = \frac{-1}{4} (x-2)^2 (x-2 + 2x + 2) = \frac{-3}{4} x(x-2) \)

\( f'(x) = 0 \iff x = 0 \text{ or } x = 2. \)

Step 3. \( f''(x) = \frac{-3}{4} [x-2 + x] = \frac{-3}{4} (2x-2) \)

\( f''(x) = 0 \iff x = 1 \text{ inflection point} \)

\( f'(0) < 0 \iff x = -1 \text{ or } x = 2 \) (double)

\( f''(0) = \frac{3}{2} > 0 \iff x = 0 \text{ is min.} \)

\( f''(2) = \frac{-3}{2} < 0 \iff x = 2 \text{ is max.} \)

\( f(0) = \frac{-1}{4} \cdot 1 \cdot (-2)^2 = -1 \)

\( f(2) = \frac{-1}{4} \cdot (2+1)(2-2)^2 = 0 \)

\( f(1) = \frac{-1}{4} \cdot (1+1)(1-2)^2 = \frac{-1}{2} \)
(b) \( f(x) = 2\sqrt{x} - \frac{x}{2} \)

Exercise!

\[ \text{Dom}(f) = [0, \infty) \]

No asymptotes

\[
\begin{align*}
\frac{f'(x)}{2\sqrt{x}} &= 0 \quad \Rightarrow \quad \sqrt{x} = 2 \quad \Rightarrow \quad x = 4 \\
f'(0) &= DNE \\
\frac{f''(x)}{2\sqrt{x}} &< 0 \quad \text{for all } x > 0
\end{align*}
\]

\[
\begin{align*}
f(x) &= \frac{4\sqrt{x} - x}{2} = 0 \quad \Rightarrow \quad 4\sqrt{x} = x \\
\Rightarrow \quad x^2 - 16x = 0 \\
x(x-16) &= 0 \\
x &= 0 \quad \text{or} \quad x = 16
\end{align*}
\]

\[
f'(4) = \frac{4 \cdot 2 - 4}{2} = 2
\]