4.6 Optimization Problems

Strategy for Solving Optimization Problems:

Step 1. Introduce variables, look for relationships among these variables, and construct a mathematical model of the form: Maximize (or minimize) \( f(x) \) on the interval \( I \).

Step 2. Find the critical values of \( f(x) \).

Step 3. Use the procedures developed in previous sections (and below) to find the absolute maximum (or minimum) value of \( f(x) \) on the interval \( I \) and the value(s) of \( x \) where this occurs.

Step 4. Use the solution to answer all questions asked in the problem.

Activity 1: A farmer has 24000 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area.

\[
\begin{align*}
x + 2y &= 24000 \quad \Rightarrow \quad y = 12000 - \frac{x}{2} \\
A &= x \cdot y \\
A(x) &= x \left( 12000 - \frac{x}{2} \right)
\end{align*}
\]

Maximize \( A(x) \) on \([0, 24000]\)

Critical numbers: \( A'(x) = 12000 - x = 0 \quad \Rightarrow \quad x = 12000 \)

\[
\begin{align*}
A(12000) &= 12000 \left( 12000 - \frac{12000}{2} \right) = 72 \times 10^6 \text{ ft}^2 \max
\\
A(0) &= 0 \text{ ft}^2
\\
A(24000) &= 0 \text{ ft}^2
\end{align*}
\]

For \( \{ x = 12000 \text{ ft} \} \) the field has the largest area.

\[
\begin{align*}
y &= 6000 \text{ ft} \\
\left( y = 12000 - \frac{x}{2} \right)
\end{align*}
\]
Activity 2: A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

\[
V = \pi r^2 h = 1 \quad \Rightarrow \quad h = \frac{1}{\pi r^2} \quad \text{for } h > 0, \quad \text{and } r > 0
\]

Surface Area: \[A(r) = 2\pi r^2 + 2\pi rh \]

\[
\frac{1}{\pi r^2} = 2\pi r^2 + \frac{2}{r}
\]

Minimize \[A(r)\]

Critical Points: \[A'(r) = 4\pi r - \frac{2}{r^2} = 0\]

\[4\pi r = \frac{2}{r^2} \quad \Rightarrow \quad r^3 = \frac{1}{2\pi} \]

\[r = \left(\frac{1}{2\pi}\right)^{\frac{1}{3}}
\]

\[h = \frac{1}{\pi r} = \frac{\left(\frac{1}{2\pi}\right)^{\frac{2}{3}}}{\pi} = 2^\frac{2}{3} \pi^{-\frac{1}{3}}
\]

Activity 3: Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola \(y = 8 - 2x^2\)

\[y = 8 - 2x^2 = 0 \quad \Leftrightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = -2, 2
\]

\[y' = -4x = 0 \quad \Rightarrow \quad x = 0
\]

\[y'' = -4 < 0 \quad \Rightarrow \quad x = 0 \text{ is a max}
\]

Area of the rectangle: \[2x(8 - 2x^2)\]

Maximize \[A(x) = 16x - 4x^3\] on \([0, 2]\)

\[A'(x) = 16 - 12x^2 = 0 \quad \Leftrightarrow \quad x^2 = \frac{4}{3} \quad \Rightarrow \quad x = \pm \frac{2}{\sqrt{3}}
\]

\[x > 0 \quad \Rightarrow \quad x = \frac{2}{\sqrt{3}}
\]

\[y = 8 - 2x^2 = 8 - 2 \cdot \frac{16}{3} = \frac{16}{3}
\]

Dimensions: \[x = \frac{2}{\sqrt{3}}, \quad y = \frac{16}{3}\]

Note: \[x = \frac{2}{\sqrt{3}}\] is a max. because \[A''(x) = -24x\]

\[A''\left(\frac{2}{\sqrt{3}}\right) < 0\]
Activity 4: Find two positive numbers whose product is 30 and whose sum is a minimum.

\[ a, b > 0 \]
\[ a \cdot b = 30 \quad b = \frac{30}{a} \]

Minimize \( a + b \)
\[ S(a) = a + \frac{30}{a} \]
\[ S'(a) = 1 - \frac{30}{a^2} = 0 \Leftrightarrow a^2 = 30 \quad a = \sqrt{30} \quad (a > 0) \]
\[ b = \frac{30}{\sqrt{30}} = \sqrt{30} \]

\[ S(\sqrt{30}) = 2\sqrt{30} \]

Activity 5: If 120 cm\(^2\) of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Surface area = \( x^2 + 4 \cdot \frac{x \cdot y}{4} = 120 \) \( \Rightarrow y = \frac{30 - \frac{x}{4}}{x} \)

Volume = \( x^2 \cdot y = x^2 \left( \frac{30 - \frac{x}{4}}{x} \right) = 30x - \frac{x^3}{4} \)

Maximize \( V(x) = 30x - \frac{x^3}{4} \)
\[ V'(x) = 30 - \frac{3x^2}{4} = 0 \Leftrightarrow x^2 = 40 \]
\[ x = \sqrt{40} \]
\[ y = \frac{30}{x} - \frac{x}{4} = \frac{30}{\sqrt{40}} - \frac{\sqrt{40}}{4} \]
\[ V''(x) = -\frac{3x}{2} \]
\[ V''(\sqrt{40}) < 0 \Rightarrow x = \sqrt{40} \text{ is a max.} \]
4.8 Antiderivatives

**Preview Activity:** Consider a function $f$ whose derivative is graphed below.

(a) On what interval(s) is $f$ an increasing function? On what intervals is $f$ decreasing?

$$f' > 0 \text{ on } (0,1.5) \cup (4,6) \Rightarrow f \text{ is increasing}$$

$$f' < 0 \text{ on } (1.5,4) \Rightarrow f \text{ is decreasing}$$

(b) On what interval(s) is $f$ concave up? concave down?

$$f'' > 0 \text{ on } (0,1) \cup (3,5) \Rightarrow f \text{ is concave up}$$

$$f'' < 0 \text{ on } (1,3) \cup (5,6) \Rightarrow f \text{ is concave down}$$

Inflection points: $x = 1, x = 3, x = 5$

(c) At what point(s) does $f$ have a relative minimum? a relative maximum?

(d) Assuming that $f(0) = 1$, sketch an estimated graph of $f$ on the grid given above.

**Definition:** A function $F$ is called an **antiderivative** of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$. 
Activity 1: Find an antiderivative of \( f(x) = x^2 \).

\[
F(x) = \frac{x^3}{3} \quad \Rightarrow \quad F'(x) = \frac{3x^2}{3} = x^2 = f(x)
\]

We see that a function has many antiderivatives. The next theorem says that all the antiderivatives of a function differ by a constant:

**Theorem:** If \( F \) is an anti-derivative of \( f \) on an interval \( I \), then the most general antiderivative of \( f \) on \( I \) is

\[
F(x) + C \quad \Rightarrow \quad F'(x) = f(x)
\]

where \( C \) is an arbitrary constant.

Activity 2: Find a particular antiderivative for the functions given in the below table.

<table>
<thead>
<tr>
<th>Function</th>
<th>Particular antiderivative</th>
<th>Function</th>
<th>Particular antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( ax )</td>
<td>( \frac{\sin x}{x} )</td>
<td>( -\cos x )</td>
</tr>
<tr>
<td>( x^n ), ( n \neq -1 )</td>
<td>( \frac{x^{n+1}}{n+1} )</td>
<td>( \sec^2 x )</td>
<td>( \tan x )</td>
</tr>
<tr>
<td>( \frac{1}{x} )</td>
<td>( \ln x )</td>
<td>( \sec x \tan x )</td>
<td>( \sec x )</td>
</tr>
<tr>
<td>( e^x )</td>
<td>( e^x )</td>
<td>( cf(x) )</td>
<td>( cF(x) )</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>( \sin x )</td>
<td>( f(x) \pm g(x) )</td>
<td>( F(x) \pm G(x) )</td>
</tr>
</tbody>
</table>

Activity 3: Find all functions \( f \) such that \( f'(x) = 4 \sin x + \frac{2x^5 - \sqrt{x}}{x} \).

\[
f'(x) = 4 \sin x + 2x^4 - x^{\frac{1}{2}}
\]

\[
f(x) = -4 \cos x + \frac{2}{5}x^5 - \left( x^{\frac{\frac{1}{2} + 1}{\frac{1}{2} + 1}} \right) = 2x^{\frac{1}{2}}
\]

\[
2\sqrt{x} \quad \Rightarrow \quad \frac{1}{x^{\frac{1}{2}}} = x^{\frac{1}{2}}
\]

\[
f(x) = -4 \cos x + \frac{2}{5}x^5 - 2x^{\frac{1}{2}} + C
\]
Activity 4: Find $f$ if $f'(x) = e^x + x^3$ and $f(0) = 3$.

$$f(x) = e^x + \frac{x^4}{4} + C$$

$$3 = f(0) = 1 + C \implies C = 2.$$ 

$$f(x) = e^x + \frac{x^4}{4} + 2$$

Activity 5: Find $f(x)$ if $f''(x) = 5 - 9x$, $f(0) = 3$, and $f(1) = 10$.

$$f'(x) = 5x - \frac{9}{2} x^2 + C$$

$$f(x) = \frac{5}{2} x^2 - \frac{9}{2} \frac{x^3}{3} + C x + d$$

$$3 = f(0) = d$$

$$f(1) = \frac{5}{2} - \frac{3}{2} + C + d = C + 4 \implies C = 6 \implies f(x) = \frac{5}{2} x^2 - \frac{3}{2} x^3 + 6x + 3$$

Activity 6: A car is traveling at 70 miles per hour when the brakes are fully applied, producing a constant deceleration of $40 \text{ ft/s}^2$. What is the distance covered before the car comes to a stop?

$$v'(t) = a(t) = -40 \text{ ft/s}^2$$

$$v(t) = 0$$

$$v(0) = 70 \ \text{m/s} = \frac{5280}{3600} = \frac{308}{3} \ \text{ ft/s}$$

$$v(t) = -40t + C$$

$$v(0) = C = \frac{308}{3}$$

$$s'(t) = v(t)$$

$$s(t) = -20t^2 + \frac{308}{3} t + d$$

$$s(0) = 0 \implies d = 0$$

$$s\left(\frac{308}{120}\right) = -20\left(\frac{308}{120}\right)^2 + \frac{308}{3} \cdot \frac{308}{120} = 132 \ \text{ft}.$$