Activity 5: Suppose that an object moving along a straight line path has its velocity $v$ (in meters per second) at time $t$ (in seconds) given by the piecewise linear function whose graph is given below. We view movement to the right as being in the positive direction (with positive velocity), while movement to the left is in the negative direction. Suppose further that the object’s initial position at time $t = 0$ is $s(0) = 1$.

![Graph of velocity and position functions]

(a) Determine the total distance traveled and the total change in position on the time interval $0 \leq t \leq 2$. What is the object’s position at $t = 2$?

- $s(0) = 1$
- $s(1) = 1 + s(0) = 2$
- $s(2) = -1 + s(1) = 1$

Total distance traveled = 2 m
Total change in position = 0 m

(b) On what time intervals is the moving object’s position function increasing? Why? On what intervals is the object’s position function decreasing? Why?

- $s$ is increasing on $(0, 1) \cup (4, 8)$ as $v > 0$
- $s$ is decreasing on $(1, 4)$ as $v < 0$
(c) What is the object’s position at $t = 8$? How many total meters has it traveled to get to this point (including distance in both directions)? Is this different from the object’s total change in position on $t = 0$ to $t = 8$?

\[
S(8) = 6 \text{ m}
\]

Total distance traveled = 13 m

Total change in position (displacement) = 5 m

(d) Find the exact position of the object at $t = 1, 2, 3, ..., 8$ and use this data to sketch an accurate graph of $y = s(t)$. How can you use the provided information about $y = v(t)$ to determine the concavity of $s$ on each relevant interval?
5.2 The Definite Integral

In the previous section we saw that a limit of the form

\[ \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i) \Delta x = \lim_{n \to \infty} \left[ f(x^*_1) \Delta x + f(x^*_2) \Delta x \ldots + f(x^*_n) \Delta x \right] \]

arises when we compute an area. We have a special terminology for such limits:

**Definition:** If \( f \) is a function defined for \( a \leq x \leq b \), we divide the interval \([a, b]\) into \( n \) subintervals of equal width \( \Delta x = (b - a)/n \). We let \( x_0(=a), x_1, \ldots, x_n(=b) \) be the endpoints of these subintervals and we let \( x^*_1, x^*_2, \ldots, x^*_n \) be any **sample points** in these intervals. Then the **definite integral of** \( f \) **from** \( a \) **to** \( b \) is

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i) \Delta x \]

provided that this limit exists. If it does exists, we say that \( f \) is **integrable** on \([a, b]\).

**Note 1:** The symbol \( \int \) is called an **integral sign** \(^1\). In the notation \( \int_a^b f(x) \, dx \), \( f(x) \) is called the **integrand** and and \( a \) and \( b \) are called the **limits of integration**; \( a \) is the **lower limit** and \( b \) is the **upper limit**. For now, the symbol \( dx \) has no meaning by itself; \( \int_a^b f(x) \, dx \) is all one symbol. The \( dx \) simply indicates that the independent variable is \( x \). The procedure of calculating an integral is called **integration**.

**Note 2:** The sum \( \sum_{i=1}^{n} f(x^*_i) \Delta x \) that occurs in the definition is called a **Riemann sum**.

\(^1\)This notation was devised by Leibniz. It is an elongated \( S \) and was chosen because an integral is a limit of sums. This notation is so useful and so powerful that its development by Leibniz must be regarded as a major milestone in the history of mathematics and science.
We know from previous section that if $f$ is positive then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1). The integral however corresponds to the area under the curve $y = f(x)$ from $a$ to $b$. (See Figure 2).

If the function $f$ takes both positive and negative values on $[a, b]$ then the integral corresponds to the net area. That is, a difference of areas:

$$\int_{a}^{b} f(x) \, dx = A_1 - A_2$$

where $A_1$ is the area of the region above the $x$-axis and below the graph of $f$, and $A_2$ is the area of the region below the $x$-axis and above the graph of $f$.

**Theorem:** If $f$ is continuous on $[a, b]$, or if $f$ has only finitely many jumps on $[a, b]$, then $f$ is integrable on $[a, b]$ (i.e. the definite integral $\int_{a}^{b} f(x) \, dx$ exists.)
If $f$ is integrable on $[a, b]$ then the definition of the integral gives the same value no matter how we choose the sample points $x_i^*$. To simplify the situation we often take the sample points to be the right endpoints. Then $x_i^* = x_i$ and the definition of the integral simplifies as follows:

**Theorem:** If $f$ is integrable on $[a, b]$, then

$$
\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x
$$

where $\Delta x = \frac{b - a}{n}$ and $x_i = a + i\Delta x$.

**Activity 1:** Express $\lim_{n \to \infty} \sum_{i=1}^{n} (x_i^2 + 3\cos x_i)\Delta x$ as an integral on the interval $[0, 1]$.

$$
\int_0^1 (x^2 + 3\cos x)\,dx
$$

**Activity 2:** Evaluate the Riemann sum for $f(x) = x^2 - x$ by taking the sample points to be right endpoints and $a = 0, b = 2$ and $n = 4$.

$$
\Delta x = \frac{b - a}{n} = \frac{2 - 0}{4} = \frac{1}{2}
$$

$$
R_4 = \sum_{i=1}^{4} f(x_i)\Delta x = \frac{1}{2} (f(x_1) + f(x_2) + f(x_3) + f(x_4))
$$

$$
= \frac{1}{2} (0 + 0 + \frac{3}{4} + 2)
$$

$$
= \frac{5}{4}
$$
Using Geometry to Evaluate Integrals: In some situations we can compute a given integral by interpreting it as the area of a familiar geometric object:

**Activity 3:** (a) Evaluate the integral \( \int_{0}^{3} (x - 1)dx \) by interpreting it as an area.

\[
y = x - 1
\]

\[
\int_{0}^{3} (x - 1)\,dx = A_2 - A_1 = 2 - \frac{1}{2} = \frac{3}{2}
\]

(b) Evaluate the integral \( \int_{0}^{1} \sqrt{1 - x^2}dx \) by interpreting it as an area.

\[
y = \sqrt{1 - x^2} \quad \text{and} \quad x^2 + y^2 = 1
\]

\[
\int_{0}^{1} \sqrt{1 - x^2} \,dx = A = \frac{1}{4} \pi (1)^2 = \frac{\pi}{4}
\]
**Midpoint Rule:** By taking the midpoint of each interval as the sample point we obtain the midpoint rule:

\[
\int_a^b f(x) \, dx \approx \sum_{i=1}^{n} f(\bar{x}_i) \Delta x
\]

where

\[
\Delta x = \frac{b - a}{n} \quad \text{and} \quad \bar{x}_i = \frac{x_{i-1} + x_i}{2} = \text{midpoint of } [x_{i-1}, x_i].
\]

**Activity 4:** Use the midpoint rule with \( n = 5 \) to approximate \( \int_0^2 \frac{1}{x} \, dx \)

\[
\Delta x = \frac{b - a}{n} = \frac{2 - 1}{5} = \frac{1}{5}
\]

\[
\int_0^2 \frac{1}{x} \, dx \approx \sum_{i=1}^{5} f(\bar{x}_i) \Delta x
\]

\[
\begin{align*}
\bar{x}_1 &= \frac{x_0 + x_1}{2} = \frac{0 + \frac{1}{5}}{2} = \frac{1}{10} \\
\bar{x}_2 &= \frac{x_1 + x_2}{2} = \frac{\frac{1}{5} + \frac{1}{10}}{2} = \frac{1}{10}
\end{align*}
\]

\[
\frac{1}{5} \sum_{i=1}^{5} f(\bar{x}_i) = \frac{1}{5} \left[ f\left(\frac{1}{10}\right) + f\left(\frac{1}{10}\right) + f\left(\frac{1}{10}\right) + f\left(\frac{1}{10}\right) + f\left(\frac{1}{10}\right) \right] = f\left(\frac{1}{10}\right) = \frac{10}{11}
\]
Properties of the Definite Integral:

- \( \int_a^b f(x) \, dx = - \int_b^a f(x) \, dx \)
- \( \int_a^a f(x) \, dx = 0 \)
- \( \int_a^b c \, dx = c(b - a) \)
- \( \int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \)
- \( \int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx \)
- \( \int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \int_a^b f(x) \, dx \)

Activity 5: Using the properties of integrals evaluate \( \int_0^1 (6 - 4x^2) \, dx \).

Activity 6: Write the following expression as a single integral

\[ \int_{-5}^{4} f(x) \, dx + \int_{4}^{8} f(x) \, dx - \int_{-5}^{6} f(x) \, dx = \]

\[ \int_{-5}^{4} f(x) \, dx + \int_{4}^{8} f(x) \, dx - \int_{-5}^{6} f(x) \, dx = \]

\[ \int_{-5}^{8} f(x) \, dx - \int_{-5}^{6} f(x) \, dx = \int_{-5}^{8} f(x) \, dx - \int_{-5}^{6} f(x) \, dx = \]

\[ \int_{-5}^{8} f(x) \, dx - \int_{-5}^{6} f(x) \, dx = \int_{-5}^{8} f(x) \, dx - \int_{-5}^{6} f(x) \, dx = \int_{-5}^{8} f(x) \, dx - \int_{-5}^{6} f(x) \, dx = \]

\[ \int_{-5}^{8} f(x) \, dx - \int_{-5}^{6} f(x) \, dx = \int_{-5}^{8} f(x) \, dx - \int_{-5}^{6} f(x) \, dx = \int_{-5}^{8} f(x) \, dx - \int_{-5}^{6} f(x) \, dx = \]
Comparison Properties of the Integral:

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$

- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

**Activity 7:** Estimate the value of the integral $\int_0^2 \frac{1}{x^2 + 1} dx$.

On $[0,2]$,

$$m = \frac{1}{5} \leq \frac{1}{x^2 + 1} \leq 1 = M$$

$$m(2-0) \leq \int_0^2 \frac{1}{x^2 + 1} dx \leq M(2-0)$$

$$\frac{2}{5} \leq \int_0^2 \frac{1}{x^2 + 1} dx \leq 2$$